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HYPERGRAPHS NOT CONTAINING A TIGHT TREE WITH A BOUNDED TRUNK*

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Abstract. An r-uniform hypergraph is a *tight r-tree* if its edges can be ordered so that every edge e contains a vertex v that does not belong to any preceding edge and the set $e - v$ lies in some preceding edge. A conjecture of Kalai personal communication published in Frankl and Füredi, *J. Combin. Theory Ser. A*, 45 (1987), pp. 226–262, generalizing the Erdős–Sós conjecture for trees, asserts that if T is a tight r-tree with t edges and G is an n-vertex r-uniform hypergraph containing no copy of T, then G has at most $\frac{t-1}{r}\binom{n}{r-1}$ edges. A *trunk* T' of a tight r-tree T is a tight subtree such that every edge of $T - T'$ has $r-1$ vertices in some edge of T' and a vertex outside T'. For $r \geq 3$, the only nontrivial family of tight r-trees for which this conjecture has been proved is the family of r-trees with trunk size one in *J. Combin. Theory Ser. A*, 45 (1987), pp. 226–262. Our main result is an asymptotic version of Kalai's conjecture for all tight trees T of bounded trunk size. This follows from our upper bound on the size of a T-free r-uniform hypergraph G in terms of the size of its shadow. We also give a short proof of Kalai's conjecture for tight r-trees with at most four edges. In particular, for 3-uniform hypergraphs, our result on the tight path of length 4 implies the intersection shadow theorem of Katona *Acta Math. Acad. Sci. Hungar.*, 15 (1964), pp. 329–337.

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1. Results and history of tight trees. In this paper, we study a Turán-type extremal problem for hypergraphs. For integers $n \geq r \geq 2$ and an r-uniform hypergraph (r-*graph*, for short) H, the *Turán number* $\mathrm{ex}_r(n, H)$ is the largest m such that there exists an n-vertex r-graph G with m edges that does not contain H. One of the well-known conjectures in extremal graph theory is the Erdős–Sós conjecture (see [2]) that *every n-vertex graph G with more than $n(t-1)/2$ edges contains every tree T with t edges as a subgraph.* In other words, they conjecture that $\mathrm{ex}_2(n, T) \leq n(t-1)/2$ for each tree with t edges. The conjecture, if true, would be best possible whenever t divides n, as seen by taking G to be the disjoint union of K_t's. There are many partial results on the conjecture. The most significant progress on the conjecture was

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made by Ajtai et al. [1], who announced a solution to the conjecture for all sufficiently large t.

In 1984, Kalai [14] made a more general conjecture for r-graphs. To describe the conjecture, we need the following notion of hypergraph trees. Let $r \geq 2$ be an integer. An r-graph T is called a tight *r-tree* if its edges can be ordered as $e_1, \dots, e_t$ so that

$$\forall i \geq 2 \ \exists v \in e_i \text{ and } 1 \leq s \leq i-1 \text{ such that } v \notin \bigcup_{j=1}^{i-1} e_j \text{ and } e_i - v \subset e_s. \tag{1}$$

Note that a graph tree is a tight 2-tree. We write $e(H)$ for the number of edges in H.

CONJECTURE 1.1 (Kalai, 1984, in [5]). *Let $r \geq 2$ and let T be a tight r-tree with $t \geq 2$ edges. Then $\mathrm{ex}_r(n, T) \leq \frac{t-1}{r}\binom{n}{r-1}$.*

A *partial (n, k, q)-Steiner system* is a family $\mathcal{F}$ of k-subsets of an n-set X such that every q-subset of X is contained in at most one member of $\mathcal{F}$. Rödl [22] showed that for all fixed $k > q \geq 2$, as $n \to \infty$, there exist partial (n, k, q)-Steiner systems of size $(1 - o(1))\binom{n}{q}/\binom{k}{q}$. Kalai observed that the r-graph obtained by taking a partial $(n, r + t - 2, r - 1)$-Steiner system $\mathcal{F}$ of maximum size and replacing each member of $\mathcal{F}$ with a complete r-graph on $r + t - 2$ vertices contains no tight r-tree with t edges and has $(\frac{t-1}{r} - o(1))\binom{n}{r-1}$ edges. Thus, Conjecture 1.1, if true, is asymptotically optimal. The same construction using the recent work of Keevash [16] (see also [11]) on the existence of designs show that in fact for every $r \geq 2$ and t there are infinitely many n for which there is an n-vertex r-graph G with $e(G) = \frac{t-1}{r}\binom{n}{r-1}$ that contains none of the tight r-trees with t edges. For example, this bound can be achieved for all $n > n_0(r, t)$ when some divisibility properties hold, e.g., $n - r + 2$ is divisible by $(t + r - 1)!$. This gives a lower bound $\frac{t-1}{r}\binom{n}{r-1} - O_{r,t}(n^{r-2})$ for all n.

A weaker upper bound

$$\mathrm{ex}_r(n, T) \leq (e(T) - 1)\binom{n}{r-1} \qquad \text{for each tight } r\text{-tree } T \tag{2}$$

is implicit in several earlier works and is explicit in [8].

PROPOSITION 1.2 (see [8, Proposition 5.4]). *Let $r \geq 2$ and T be a tight r-tree with t edges. If G is a T-free r-graph on n-vertices, then $e(G) \leq (t-1)|\partial(G)| \leq (t-1)\binom{n}{r-1}$, where $\partial(G) = \{S \subseteq V(G) : |S| = r - 1 \text{ and } S \subseteq e \text{ for some } e \in E(G)\}$.*

To prove Conjecture 1.1, we need to improve the bound in (2) by a factor of r. This turns out to be difficult even for very special cases of tight trees. It is only recently that the authors [9] were able to improve the bound in (2) by roughly a factor of 2 in the case where T is the tight r-uniform path with t edges. (For short paths, $t < (3/4)r$, Patkós [21] proved better coefficients. Detailed calculations are available at http://www.renyi.hu/~patkos/tight-paths-fixed.pdf.)

So far, the only family of tight trees for which Kalai's conjecture is verified is the family of so-called star-shaped trees. A tight r-tree T is star-shaped if it contains an edge e_0 such that $|e \cap e_0| = r - 1$ for each $e \in T \setminus \{e_0\}$.

THEOREM 1.3 (see [5]). *Let $n, r, t \geq 2$ be integers. Let G be an n-vertex r-graph with $e(G) > \frac{t-1}{r}\binom{n}{r-1}$. Then G contains every star-shaped tight r-tree with t edges.*

Given a tight r-tree T and a tight subtree T' of T, we say that T' is a *trunk* of T if there exists an edge-ordering of T satisfying (1) such that the edges of T' are listed first and for each $e \in E(T) \setminus E(T')$ there exists $e' \in E(T')$ such that $|e \cap e'| = r - 1$. Let $c(T)$ be the minimum number of edges in a trunk of T. Hence, a star-shaped tight tree is a tight tree T with $c(T) = 1$, and Theorem 1.3 says that Kalai's conjecture holds for tight r-trees T with $c(T) = 1$. Note from the definition above that for a tight tree T having $c(T) \le c$ is equivalent to saying that all but at most c edges of T contain a vertex of degree 1.

The primary goal of this paper is to extend Theorem 1.3 to tight trees of bounded trunk size. Our main theorem says that for all fixed integers $r \ge 2$ and $c \ge 1$, Kalai's conjecture holds asymptotically in $e(T)$ for tight r-trees T with $c(T) \le c$.

THEOREM 1.4. *Let n, r, t, c be positive integers, where $n \ge r \ge 2$ and $t \ge c \ge 1$. Let $a(r, c) = (r^r + 1 - \frac{1}{r})(c - 1)$. Let T be a tight r-tree with t edges and $c(T) \le c$. Then*

$$\mathrm{ex}_r(n, T) \le \left(\frac{t-1}{r} + a(r, c)\right)\binom{n}{r-1}. \tag{3}$$

Note that Theorem 1.3 follows from Theorem 1.4 by setting $c = 1$. The main point of Theorem 1.4 is that the coefficient in front of $\binom{n}{r-1}$ is $(t-1)/r + O_{r,c}(1)$, while the coefficient in Kalai's conjecture is $(t-1)/r$.

We also give a (simple) proof of the fact that Kalai's conjecture holds for tight r-trees with at most four edges.

THEOREM 1.5. *Let $n \ge r \ge 2$ be integers and T be a tight r-tree with $t \le 4$ edges. Then*

$$\mathrm{ex}_r(n, T) \le \frac{t-1}{r}\binom{n}{r-1}.$$

The proofs of (stronger versions of) Theorems 1.4 and 1.5 are postponed to sections 4 and 5.

2. Tight trees and shadows. An important notion in extremal set theory is that of a shadow. Given an r-graph G, the *shadow* of G is

$$\partial(G) = \{S : |S| = r - 1, \quad \textit{and} \quad S \subseteq e \quad \textit{for some} \quad e \in E(G)\}.$$

In fact, Frankl and Füredi [5] proved the following stronger version of Theorem 1.3.

THEOREM 2.1 (see [5]). *If T is any star-shaped tight r-tree with t edges and G is a T-free r-graph, then $e(G) \le \frac{t-1}{r}|\partial(G)|$.*

There were several other results in the literature that bound the size of an H-free r-graph in terms of the size of its shadow. One of the first results of this kind is the following *intersection shadow theorem* of Katona. An r-graph G is *intersecting* if every two edges of it intersect, i.e., if G contains no matching of size two.

THEOREM 2.2 (see [15]). *Let $r \ge 2$. If G is an intersecting r-graph, then $e(G) \le |\partial(G)|$.*

More recently, Frankl [4] showed that if G is an r-graph that does not contain a matching of size $s + 1$, then $e(G) \le s|\partial(G)|$. Sometimes it is easier to prove the bounds in terms of the shadow size than in terms of n using induction. Instead of Theorems 1.4 and 1.5 we will prove bounds on $e(G)$ in terms of $|\partial(G)|$ in Theorems 4.1 and 5.1, from which Theorems 1.4 and 1.5 will follow.

Based on our results, we propose the following conjecture, which we will show is equivalent to Kalai's conjecture.

CONJECTURE 2.3. *Let $r \ge 2, t \ge 1$ be integers. Let T be a tight r-tree with t edges. If G is an r-graph that does not contain T, then $e(G) \le \frac{t-1}{r}|\partial(G)|$.*

The lower bound constructions obtained from designs mentioned earlier show that the bound in Conjecture 2.3, if true, would be tight. Since for every r-graph G on n vertices one has $|\partial(G)| \le \binom{n}{r-1}$ Conjecture 2.3 obviously implies Conjecture 1.1. We will show in Theorem 2.4 that Conjecture 1.1 also implies Conjecture 2.3.

THEOREM 2.4. *If T is a tight tree, then the limit*

$$\alpha(T) := \lim_{n\to\infty} \mathrm{ex}_r(n,T)/\binom{n}{r-1}$$

exists and is equal to its supremum. Moreover,

$$\alpha(T) = \sup\left\{\frac{e(G)}{|\partial(G)|} : G \textit{ is a } T\textit{-free } r\textit{-graph}\right\}.$$

In particular for $\alpha := \alpha(T)$ we have $\mathrm{ex}_r(n,T) \le \alpha\binom{n}{r-1}$ and $e(G) \le \alpha|\partial(G)|$ for every n and for every T-free r-graph G.

To prove Theorem 2.4, we need another result from the literature. Let $n \ge k \ge q \ge 1$. Let H be a q-uniform hypergraph on k vertices. A (n,k,H)-*packing* of size m is a collection $\{H_1,\dots,H_m\}$ of copies of H with vertex sets $V_1,\dots V_m$, respectively, such that with $V := \bigcup_{i=1}^m V_i$ we have $|V| \le n$ and each q-set in V is an edge of at most one H_i. Note that when H is the complete q-graph on k-vertices, an (n,k,H)-packing is equivalent to a partial (n,k,q)-Steiner system mentioned in the introduction. Clearly, an (n,k,H)-packing has size at most $\binom{n}{q}/e(H)$. Generalizing Rödl's result [22] mentioned earlier, Frankl and Füredi [6] proved for any given q-graph H on k vertices, as $n\to\infty$, there exist (n,k,H)-packings of size $(1-o(1))\binom{n}{q}/e(H)$.

LEMMA 2.5. *Let T be a tight r-tree and suppose that G is a T-free r-graph. Then for every $\varepsilon > 0$, there exists $n_0 = n_0(T,G,\varepsilon)$ such that for all $n > n_0$*

$$\mathrm{ex}_r(n,T) > \left(\frac{e(G)}{|\partial(G)|} - \varepsilon\right)\binom{n}{r-1}.$$

Proof of Lemma 2.5. Let G be the given T-free r-graph. Let $H = \partial(G)$. Then H is an $(r-1)$-graph on $k := n(G)$ vertices. By the abovementioned packing result of Frankl and Füredi [6], there exists an (n,k,H)-packing $H_1,\dots,H_m$ with vertex sets $V_1,\dots,V_m$ such that $m = (1-o(1))\binom{n}{r-1}/e(H)$. For each $i \in [m]$, let G_i be a copy of G on V_i such that $\partial(G_i) = H_i$. By our definition, $\partial(G_1),\dots,\partial(G_m)$ are pairwise disjoint and hence in particular $G_1,\dots,G_m$ are pairwise edge-disjoint. Let $F = \bigcup_{i=1}^m G_i$. Then F is an r-graph on at most n vertices that has $(1-o(1))((e(G)/|\partial(G)|)\binom{n}{r-1}$ edges.

It remains to show that F is T-free. Suppose $e_1,\dots,e_t$ is an ordering of the edges of T that satisfies (1). Suppose F contains a copy T' of T with e_1 mapped to an edge e'_1 of G_i for some $i \in [m]$. Since $\partial(G_j)\cap\partial(G_i) = \emptyset$ for all $j \ne i$, no edge in $E(F)\setminus E(G_i)$ intersects any edge of G_i in an $(r-1)$-set. This forces all the edges of T' to lie in G_i, contradicting G_i being a T-free. $\square$

Lemma 2.5 may be viewed as a generalization of Kalai's construction mentioned in the introduction. Similar ideas as the one used in the proof of Lemma 2.5 have been used in various earlier works. See [13] for another application of such an idea.

Remark 2.6. It follows from the proof of Lemma 2.5 that the lemma still holds if T is replaced with any r-graph with a *connected* $(r-1)$-intersection graph, meaning that the auxiliary graph defined on $E(T)$ where $e,e' \in E(T)$ are adjacent if and only if $|e\cap e'| = r-1$ is connected.

Proof of Theorem 2.4. Define

$$\alpha(n,T) := \mathrm{ex}_r(n,T)/\binom{n}{r-1},$$
$$\beta(n,T) := \max\left\{\frac{e(G)}{|\partial G|} :\ G \text{ is a } T\text{-free } r\text{-graph on } n \text{ vertices}\right\}.$$

Since $\beta(n,T) \le \beta(n+1,T)$ and $\beta(n,T) \le e(T)-1$ (by Proposition 1.2) the limit $\beta = \beta(T) = \lim_{n\to\infty}\beta(n,T)$ exists, is positive, and is equal to its supremum. Since $\alpha(n,T) \le \beta(n,T)$ we have $\sup_n \alpha(n,T) \le \beta$. The proof of the theorem can be completed by applying Lemma 2.5. Indeed, for every $\varepsilon > 0$, we can take a T-free r-graph G with $\frac{e(G)}{|\partial(G)|} > \beta - \varepsilon$. By Lemma 2.5 there exists an n_0 such that $\alpha(n,T) > \beta - 2\varepsilon$ for all $n > n_0$. ∎

As for a corollary of Theorem 2.4, we have the following.

COROLLARY 2.7. *Conjecture* 2.3 *is equivalent to Kalai's conjecture.*

3. Preliminary lemmas. Given an r-graph G and a subset $D \subseteq V(G)$, we define the *link* of D in G, denoted by $L_G(D)$, to be

$$L_G(D) = \{e \setminus D : e \in E(G), D \subseteq e\}.$$

The *degree* of D, denoted by $d_G(D)$, is defined to be $|L_G(D)|$; equivalently it is the number of edges of G that contain D. When G is r-uniform and $|D| = r-1$, elements of $L_G(D)$ are vertices. In this case, we also use $N_G(D)$ to denote $L_G(D)$ and call it the *co-neighborhood* of D in G. When the context is clear we will drop the subscripts in $L_G(D)$, $N_G(D)$, and $d_G(D)$. For each $1 \le p \le r-1$, we define the *minimum nonzero p-degree* of G to be

$$\delta_p(G) = \min\{d_G(D) : |D| = p, \quad \text{and} \quad D \subseteq e \quad \text{for some} \quad e \in E(G)\}.$$

Hence, by this definition, each p-set D in G either has degree 0 or has degree at least $\delta_p(G)$. Given an r-graph G, and $D \in \partial(G)$, let $w(D) = \frac{1}{d_G(D)}$. For each $e \in E(G)$, let

$$w(e) = \sum_{D\in\binom{e}{r-1}} w(D) = \sum_{D\in\binom{e}{r-1}} \frac{1}{d_G(D)}. \tag{4}$$

We call w the *default weight function* on $E(G)$ and $\partial(G)$. The following simple property of the default weight function is key to the weight method, employed in [5] and in various other works.

LEMMA 3.1. *Let G be an r-graph. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then*

$$\sum_{e\in E(G)} w(e) = |\partial(G)|.$$

Proof. For convenience, let $E = E(G)$. By definition,

$$\sum_{e\in E} w(e) = \sum_{e\in E}\sum_{D\in\binom{e}{r-1}} \frac{1}{d_G(D)} = \sum_{D\in\partial(G)}\sum_{D\subseteq e\in E}\frac{1}{d_G(D)} = \sum_{D\in\partial(G)} 1 = |\partial(G)|.$$
∎

An r-graph G is called *r-partite* if $V(G)$ can be partitioned into r sets $A_1, \dots, A_r$ such that every edge of G contains one vertex from each A_i. We call $(A_1, \dots, A_r)$ an *r-partition* of G. Equivalently, we say that an r-graph G is *r-colorable* if there exists a vertex coloring of G with r colors such that each edge uses all r colors; we call such a coloring a *proper r-coloring* of G. The following proposition follows by induction on the number of edges in T.

PROPOSITION 3.2. *Let $r \geq 2$. Every tight r-tree T has a unique r-partition.*

Given r-graphs G and H, an *embedding* of H into G is an injection $f : V(H) \to V(G)$ such that for each $e \in E(H)$, $f(e) \in E(G)$.

LEMMA 3.3 (color-preserving embedding). *Let T be a tight r-tree with t edges. Let φ be a proper r-coloring of T. Let G be an r-partite graph with $\delta_{r-1}(G) \geq t$, and let $(A_1, \dots, A_r)$ be an r-partition of G. Then there exists an embedding f of T into G such that for each $u \in V(T)$ $f(u) \in A_{\varphi(u)}$.*

Proof. We use induction on t. The base step is trivial. Now, suppose $t \geq 2$. Let $e_1, \dots, e_t$ be an ordering of the edges of T that satisfies (1). Let $T' = T \setminus e_t$. Then T' is a tight r-tree with $t-1$ edges. By the induction hypothesis, there exists an embedding f of T' into G such that for each $u \in V(T')$, $f(u) \in A_{\varphi(u)}$. By the definition of T, there exists an edge $e_{\alpha(t)} \in E(T')$ such that $|e_t \cap e_{\alpha(t)}| = r-1$. Let $D = e_t \cap e_{\alpha(t)}$ and let v be the unique vertex in $e_t \setminus e_{\alpha(t)} = V(T) \setminus V(T')$. Then $e_t = D \cup \{v\}$. Since $f(D)$ is an $(r-1)$-set contained in $f(e_{t-1})$ and $\delta_{r-1}(G) \geq t$, $d_G(f(D)) \geq t$. So there are at least t edges of G containing $f(D)$. At most $t-1$ of them contain $f(D)$ and a vertex in $f(T') \setminus f(D)$. So there exists an edge e in G that contains $f(D)$ and a vertex z outside $f(T')$. We extend f by letting $f(v) = z$. Now f is an embedding of T into G.

It remains to show that $z \in A_{\varphi(v)}$. By permuting colors if needed, we may assume that $\varphi(v) = r$. Since $D \cup \{v\} \in E(T)$ and φ is proper, the colors used in D are $1, \dots, r-1$. By our assumption, vertices in $f(D)$ lie in $A_1, \dots, A_{r-1}$, respectively, which implies $z \in A_r$. ∎

The following lemma is folklore. We include a proof for completeness. Recall that given an r-graph G, $\partial(G) = \{S : |S| = r-1, \text{ and } S \subseteq e \text{ for some } e \in E(G)\}$ and $\delta_{r-1}(G) = \min\{d_G(D) : D \in \partial(G)\}$.

LEMMA 3.4. *Let $r \geq 2$ and $q \geq 1$ be integers and let G be an r-graph with $e(G) > q|\partial(G)|$. Then G contains a subgraph G' with $\delta_{r-1}(G') \geq q+1$ and*

$$e(G') > q|\partial(G')|. \tag{5}$$

Proof. Among subgraphs G' of G satisfying (5), choose one with the fewest edges. We claim that $\delta_{r-1}(G') \geq q+1$. Indeed, if there is $D \in \partial(G')$ that is contained in at most q edges of G', then the r-graph G'' obtained from G' by deleting all edges containing D again satisfies (5) but has fewer edges than G', a contradiction. ∎

Another useful folklore fact is the following.

LEMMA 3.5. *Let α be a positive real, $r \geq 3$ be an integer, and G be an r-graph with $e(G) > \frac{\alpha}{r}|\partial(G)|$. Then there is $v \in V(G)$ such that the link $G_1 := L_G(\{v\})$ satisfies*

$$e(G_1) > \frac{\alpha}{r-1}|\partial(G_1)|.$$

Proof. Suppose that $|L_G(\{v\})| \leq \frac{\alpha}{r-1}|\partial(L_G(\{v\})|$ for each $v \in V(G)$. Then

$$r \cdot e(G) = \sum_{v\in V(G)} d_G(v) = \sum_{v\in V(G)} |L_G(\{v\})| \leq \frac{\alpha}{r-1} \sum_{v\in V(G)} |\partial(L_G(\{v\})|.$$

Since each edge $f \in \partial(G)$ contributes $r-1$ to $\sum_{v\in V(G)} |\partial(L_G(\{v\})|$ (1 to the link of each of its vertices), this proves the lemma. ∎

We also need the following fact used in [5].

Proposition 3.6. *Let r be a positive integer. Let $d_1 \leq d_2, \cdots \leq d_r$ be positive reals. If $\sum_{i=1}^r \frac{1}{d_i} = s$, then for each $i \in [r]$, $d_i \geq \frac{i}{s}$.*

Proof. For each $i \in [r]$, since $\frac{1}{d_1} \geq \cdots \geq \frac{1}{d_i}$, we have $\frac{i}{d_i} \leq \sum_{j=1}^i \frac{1}{d_j} \leq s$. So, $d_i \geq \frac{i}{s}$. ∎

4. Proof of Theorem 1.4 on trees with bounded trunks. As discussed in section 2, we prove the following stronger version of Theorem 1.4, from which Theorem 1.4 follows immediately. Recall that given a tight r-tree T, $c(T)$ is the small m such that T contains a tight subtree T' with m edges and each edge in $E(T)\setminus E(T')$ contains $r-1$ vertices in some edge of T' and a vertex outside T'.

Theorem 4.1. *Let n, r, t, c be positive integers, where $n \geq r \geq 2$ and $t \geq c \geq 1$. Let $a(r,c) = (r^r + 1 - \frac{1}{r})(c-1)$. Let T be a tight r-tree with t edges and $c(T) \leq c$. If G is an r-graph that does not contain T, then*

$$e(G) \leq \left(\frac{t-1}{r} + a(r,c)\right) |\partial(G)|. \tag{6}$$

Proof. Suppose T is a tight r-tree with t edges and $c(T) = c$. Let G be an n-vertex r-graph with $e(G) > (\frac{t-1}{r} + a(r,c))|\partial(G)|$. We show that G contains T. For convenience, let

$$\gamma = \frac{t-1}{r} + a(r,c) - r^r(c-1) = \frac{t-1}{r} + \left(1 - \frac{1}{r}\right)(c-1). \tag{7}$$

Then

$$e(G) > (\gamma + r^r(c-1))|\partial(G)|.$$

Let w be the default weight function on $E(G)$ and $\partial(G)$. By Lemma 3.1,

$$\sum_{e\in E(G)} w(e) = |\partial(G)|.$$

Let

$$H = \left\{e \in E(G) : w(e) \geq \frac{1}{\gamma}\right\} \text{ and } L = \left\{e \in E(G) : w(e) < \frac{1}{\gamma}\right\}.$$

By the definition of H,

$$\frac{1}{\gamma} e(H) \leq \sum_{e\in H} w(e) \leq \sum_{e\in G} w(e) = |\partial(G)|.$$

Hence $e(H) \leq \gamma|\partial(G)|$. Since $e(G) > (\gamma + r^r(c-1))|\partial(G)|$, we have

$$e(L) > r^r(c-1)|\partial(G)|.$$

A random r-coloring of $V(L)$ yields that L contains an r-partite subgraph L_1 with $e(L_1) \geq \frac{r!}{r^r} e(L)$. Then

$$e(L_1) > \frac{r!}{r^r} r^r (c-1)|\partial(G)| = r!(c-1)|\partial(G)|. \tag{8}$$

Let $(A_1, \ldots, A_r)$ be an r-partition of L_1. Let $e \in E(L_1)$. Let σ be a permutation of $[r]$ such that

$$d_G(e \setminus A_{\sigma(1)}) \leq \cdots \leq d_G(e \setminus A_{\sigma(r)}).$$

We let $\pi(e) = (\sigma(1), \ldots, \sigma(r))$ and refer to it as the *pattern* of e. Since there are $r!$ different permutations of $[r]$, by the pigeonhole principle, some $\lceil e(L_1)/r! \rceil$ edges e of L_1 have the same pattern $\pi(e)$. Let L_2 be the subgraph of L_1 consisting of these edges. By (8),

$$e(L_2) \geq \frac{e(L_1)}{r!} > (c-1)|\partial(G)|.$$

By Lemma 3.4, L_2 contains a subgraph L_2^* such that

$$\delta_{r-1}(L_2^*) \geq c.$$

Recall that all edges in $L_2^* \subseteq L_1$ have the same pattern. By permuting indices if needed, we may assume that $\pi(e) = (1, 2, \ldots, r)$ for each $e \in L_2^*$. By our assumption,

$$d_G(e \setminus A_1) \leq \cdots \leq d_G(e \setminus A_r) \qquad \forall e \in L_2^*. \tag{9}$$

Also, by the definition of L,

$$w(e) = \sum_{i=1}^{r} \frac{1}{d_G(e \setminus A_i)} < \frac{1}{\gamma} \qquad \forall e \in L_2^* \subseteq L.$$

By Proposition 3.6 and (9), we have

$$d_G(e \setminus A_i) > i\gamma \qquad \forall e \in L_2^* \qquad \forall i \in [r]. \tag{10}$$

Now consider a trunk T' of T with c edges. By the definition of a trunk, if E' is any subset of $E(T) \setminus E(T')$, then $T' \cup E'$ is a tight tree with $c + |E'|$ edges. By Proposition 3.2, T' is r-partite. Let $(B_1, \ldots, B_r)$ be an r-partition of T'. For each $e \in E(T) \setminus E(T')$, by definition, there exists $\alpha(e) \in E(T')$ such that $|e \cap \alpha(e)| = r-1$. Thus, $e \cap \alpha(e) = \alpha(e) \setminus B_i$ for some unique $i \in [r]$. For each $i \in [r]$, let

$$E_i = \{e \in E(T) \setminus E(T') : e \cap \alpha(e) = \alpha(e) \setminus B_i\}.$$

By permuting the subscripts in the r-partition $(B_1, \ldots, B_r)$ of T' if needed, we may assume that

$$|E_1| \leq \cdots \leq |E_r|.$$

Since $\sum_{i=1}^{r} |E_i| = t - c$, this implies

$$|E_1| + \cdots + |E_i| \leq \left\lfloor \frac{i(t-c)}{r} \right\rfloor \qquad \forall i \in [r]. \tag{11}$$

Since $e(T') = c$, $\delta_{r-1}(L_2^*) \geq c$, $(A_1, \ldots, A_r)$ is an r-partition of L_2^*, and $(B_1, \ldots, B_r)$ is an r-partition of T', by Lemma 3.3, there exists an embedding h of T' into L_2^* such that for each $i \in [r]$ every vertex in B_i of T' is mapped into A_i. Let $i \in [r] \setminus \{1\}$ and suppose that we have extended h to an embedding of $T' \cup E_1 \cup \cdots \cup E_{i-1}$ into G. By the definition of E_i, for each $e \in E_i$ there is $\alpha(e) \in T'$ such that $e \cap \alpha(e) = \alpha(e) \setminus B_i$ and $h(e \cap \alpha(e)) = h(\alpha(e)) \setminus A_i$. By (10),

$$d_G(h(e \cap \alpha(e)) \geq \lfloor i\gamma \rfloor + 1 \qquad \forall e \in E_i. \tag{12}$$

Since $T' \cup E_1 \cup \cdots \cup E_i$ is a tight tree with

$$c + |E_1| + \cdots + |E_i| \leq \lfloor \frac{i(t-c)}{r} \rfloor + c \leq \lfloor i\gamma \rfloor + 1$$

edges, where the two inequalities follows from (11) and (7), respectively, and h is already an embedding of $T' \cup E_1 \cup \cdots \cup E_{i-1}$ into G, (12) ensures that we can greedily extend h further to an embedding of $T' \cup E_1 \cup \cdots \cup E_i$ into G. Hence we can find an embedding of T into G. ∎

5. Proof of Theorem 1.5 on trees with four edges.

We prove the following shadow version of Theorem 1.5 from which Theorem 1.5 follows immediately.

THEOREM 5.1. *Let $n \geq r \geq 2$ be integers and T be a tight r-tree with $t \leq 4$ edges. If G is an r-graph that does not contain T, then $e(G) \leq \frac{t-1}{r}|\partial(G)|$.*

We start from a special case of such T, the 3-uniform tight path P_4^3 with four edges. The case of the path P_5^3 is still unsolved (to our knowledge).

LEMMA 5.2. *Let $n \geq 5$ and G be an n-vertex 3-graph containing no tight path P_4^3 with four edges. Then $e(G) \leq |\partial(G)|$.*

Observe that for 3-graphs Lemma 5.2 is stronger than Katona's intersecting shadow theorem, since an intersecting 3-graph must be P_4^3-free. There are many nearly extremal families with very different structures for Lemma 5.2 besides the ones obtained from Steiner systems $S(n, 5, 2)$. Here we mention just two. First, one can observe that the Erdős–Ko–Rado family $G := \{g \in \binom{[n]}{3} : 1 \in g\}$ is P_4^3-free with

$$|\partial(G)| = \binom{n}{2} = \frac{n}{n-2}\binom{n-1}{2} = \frac{n}{n-2}e(G).$$

Second, for $n \equiv 0 \mod 3$ one can take a tournament $\overrightarrow{D}$ on $n/3$ vertices and a partition of $[n]$ into triples $V_1, V_2, \ldots, V_{n/3}$ and define the P_4^3-free triple system as

$$G := \left\{ g \in \binom{[n]}{3} : \text{ for some } \overrightarrow{ij} \in E(\overrightarrow{D}) \text{ one has } |V_i \cap g| = 2, |V_j \cap g| = 1 \right\}.$$

Then we have $|\partial(G)|/e(G) = \binom{n}{2}/9\binom{n/3}{2} = (n-1)/(n-3)$.

Proof of Lemma 5.2. Suppose G is an n-vertex 3-graph with the fewest edges such that

$$e(G) > |\partial(G)| \text{ and } G \text{ contains no } P_4^3. \tag{13}$$

By Lemma 3.4 and the minimality of G,

$$\delta_2(G) \geq 2. \tag{14}$$

Let w be the default weight function on G and $\partial(G)$. By Proposition 3.1, $\sum_{e \in G} w(e) = |\partial(G)|$. Since $e(G) > |\partial(G)|$ by (13), G has an edge $e_0 = abc$ with

$$w(e_0) = \frac{1}{d(ab)} + \frac{1}{d(ac)} + \frac{1}{d(bc)} < 1. \tag{15}$$

We may assume $d(ab) \le d(ac) \le d(bc)$. Similarly to Proposition 3.6, in order for (15) to hold, we need

$$d(ac) \ge 3 \quad \text{and} \quad d(bc) \ge 4. \tag{16}$$

By (14) and (16), we can greedily choose distinct $a', b', c' \in V(G) - \{a, b, c\}$ so that $abc', acb', bca' \in G$.

We claim that

$$ab'b, ac'c \in G. \tag{17}$$

Indeed, by (14) G has an edge $ab'x$ for some $x \ne c$. If $x \notin \{b, a'\}$, then G has a tight 4-path $a'bcab'x$, a contradiction to (13). So suppose $x = a'$. By (16), G has an edge bcy for some $y \notin \{a, a', b'\}$. Then G has a tight 4-path $ybcab'a'$, again a contradiction to (13). Thus $ab'b \in G$. Similarly, $ac'c \in G$, and (17) holds.

Next we similarly show that

$$a'ba, a'ca \in G. \tag{18}$$

Indeed, by (14) G has an edge $a'bx$ for some $x \ne c$. If $x \notin \{a, b'\}$, then G has a tight 4-path $b'acba'x$. Suppose $x = b'$. Then by (17), G has a tight 4-path $b'a'bcac'$, again a contradiction to (13). Thus $a'ba \in G$. Similarly, $a'ca \in G$, and (18) holds.

Together, (17) and (18) imply that $d_G(ab) \ge 4$ and $d_G(ac) \ge 4$. So, the proof of (17) yields similarly that $c'bc, b'cb \in G$. If the degree of each of $a'a, a'b, a'c$ is 2, then the 3-graph $G_2 = G \setminus \{a'ab, a'ac, a'bc\}$ has $|G|-3$ edges and $|\partial(G_2)| = |\partial(G)|-3$, a contradiction to the minimality of G. Thus we may assume that G has an edge $a'ax$, where $x \notin \{b, c\}$. By the symmetry between b' and c', we may assume $x \ne b'$. Then G has a tight 4-path $xa'abcb'$. ◻

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. We use induction on r. The base step of $r = 2$ follows from the fact that the Erdős–Sós conjecture has been verified for all trees of diameter at most four [20].

For the induction step, suppose $r \ge 3$ and that the theorem holds for all $r' < r$, T is a tight r-tree. Let G be an r-graph with $e(G) > \frac{t-1}{r}|\partial(G)|$.

Case 1. T has a vertex v belonging to all edges. Let T_1 be the link $L_T(\{v\})$ of v. It is a tight $(r-1)$-tree with t edges. By Lemma 3.5, there is $a \in V(G)$ such that the link $G_1 := L_G(\{a\})$ satisfies $e(G_1) > \frac{t-1}{r-1}|\partial(G_1)|$. By the induction assumption, there is an embedding φ of T_1 into G_1. Then by letting $\varphi(v) = a$ we obtain an embedding of T into G.

Case 2. T has no vertex belonging to all edges. By the definition of a tight r-tree, this is possible only if $t = 4$, $r = 3$, and $T = P_4^3$. In this case, we are done by Lemma 5.2. ◻

6. Concluding remarks.

- Theorem 2.4 shows that some shadow theorems in the literature are not really stronger than their nonshadow versions. In particular, this is the case whenever the forbidden r-graph T has a connected $(r-1)$-intersection graph (see Remark 2.6).
- It would be interesting to decide if Lemma 2.5 holds for other r-graphs besides tight trees and also for which r-graphs T $\lim_{n\to\infty} \mathrm{ex}_r(n,T)/\binom{n}{r-1}$ exists. In particular, we ask if $\lim_{n\to\infty} \mathrm{ex}_r(n,T)/\binom{n}{r-1}$ exists for each r-uniform forest

T, where an r-graph is a *forest* if it is a subgraph of a tight tree. This question is not even solved when $r = 2$ and T is a graph forest; see, e.g., [19]. See [8] and [18] for recent results on the Turán numbers of some large families of r-uniform forests.

Note that even for forests, if the limits $\alpha(T)$ and $\beta(T)$ exist they need not be equal, where as in Theorem 2.4 and its proof, $\alpha(T) = \lim_{n\to\infty} \mathrm{ex}_r(n, T)/\binom{n}{r-1}$ and

$$\beta(T) = \lim_{n\to\infty} \max\{e(G)/|\partial(G)| : G \text{ is an } n\text{-vertex } T\text{-free } r\text{-graph}\}.$$

Consider an r-uniform linear path L_4^r of length four, $E(L_4^r) := \{\{1, 2, \ldots, r\}, \{r, r+1, \ldots, 2r-1\}, \{2r-1, 2r, \ldots, 3r-2\}, \{3r-2, 3r-1, \ldots, 4r-3\}\}$. It is known [8, 18] that $\mathrm{ex}(n, L_4^r) = \binom{n-1}{r-1} + \binom{n-3}{r-2} + \varepsilon(n, r)$ for $n > n_0(r)$ and $r \geq 3$, where $\varepsilon(n, r) = 0$ except for $r = 3$, when it is 0, 1, or 2. So we have $\alpha(L_4^r) = 1$. On the other hand, the complete r-graph G on $4r - 4$ vertices avoids L_4^r and $e(G)/|\partial G| = \binom{4r-4}{r}/\binom{4r-4}{r-1} = (3r-3)/r \leq \beta(P)$. Consequently, $0 < \alpha(L_4^r) < \beta(L_4^r)$ for $r \geq 3$.

In the case $r = 2$ consider $T = kP_2$, a disjoint union of k paths of length 2 on $3k$ vertices. Gorgol [12] showed that $\alpha(kP_2) = k - 1/2$, while considering the complete graph on $3k-1$ vertices we get $\beta(kP_2) \geq (3k-2)/2$. Moreover, the Erdős–Gallai theorem implies that here equality holds.

- Recent substantial work by Keller and Lifshitz [17] studies the Turán number of some r-graphs F with small core. However, their *junta method for hypergraphs* does not seem to apply here, since it seems to require that $r \gg |C|$, where C is the set of the vertices of F of degree at least 2.
- In a subsequent manuscript [10], we are able to sharpen the result in this paper for 3-uniform tight trees T with $c(T) = 2$ and show that Kalai's conjecture holds for these tight trees T with at least 20 edges.

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