SIAM J. DISCRETE MATH.
Vol. 26, No. 3, pp. 1238–1255

TURÁN NUMBERS OF SUBDIVIDED GRAPHS*

TAO JIANG† AND ROBERT SEIVER†

Abstract. Given a positive integer n and a graph F, the *Turán number* $ex(n, F)$ is the maximum number of edges in an n-vertex simple graph that does not contain F as a subgraph. Let H be a graph and p a positive even integer. Let $H^{(p)}$ denote the graph obtained from H by subdividing each of its edges $p-1$ times. We prove that $ex(n, H^{(p)}) = O(n^{1+(16/p)})$. This follows from a more general result that we establish, where different edges of H are allowed to be subdivided different numbers of times. Our result is closely related to the results of Jiang [*J. Graph Theory*, 67 (2011), pp. 139–152] and of Kostochka and Pyber [*Combinatorica*, 8 (1988), pp. 83–86] on topological minors.

Key words. Turán number, subdivision, subdivided graph, topological clique, minor

AMS subject classifications. 05C35, 05C83, 05C42

DOI. 10.1137/100819254

1. Introduction. We consider only simple graphs in this paper unless otherwise specified. In extremal graph theory, we are typically interested in studying thresholds on edge density beyond which certain substructures are forced to appear. The well-known Turán problem is one of this kind. Given a family $\mathcal{F}$ of graphs and a positive integer n, the Turán number $ex(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an n-vertex graph not containing any member of $\mathcal{F}$ as a subgraph. Hence, if an n-vertex graph G has more than $ex(n, \mathcal{F})$ edges, then it must contain some member of $\mathcal{F}$ as a subgraph.

Let $\mathcal{F}$ be a family of graphs, and let $p = \min\{\chi(F) : F \in \mathcal{F}\}$. The celebrated Erdős–Stone–Simonovits theorem says that $ex(n, \mathcal{F}) = (1 - \frac{1}{p-1})\binom{n}{2} + o(n^2)$, where $p = \min\{\chi(F) : F \in \mathcal{F}\}$. This determines $ex(n, \mathcal{F})$ asymptotically when $\mathcal{F}$ consists solely of nonbipartite graphs. When $\mathcal{F}$ contains a bipartite graph, however, our knowledge about $ex(n, \mathcal{F})$ is quite limited with only a few exceptions, most notably that concerning $ex(n, \{C_4\})$, where the exact value is determined for infinitely many values of n. As a starting point, it is natural to focus on $ex(n, \mathcal{F})$ when $\mathcal{F}$ consists of a single graph F, in which case, we will write $ex(n, F)$ for $ex(n, \{F\})$.

The determination of $ex(n, F)$ for a bipartite graph F turns out to be very difficult. It is known that there are positive constants c_1, c_2 depending on F such that $\Omega(n^{1+c_1}) \leq ex(n, F) \leq O(n^{2-c_2})$. However, only for very few bipartite graphs F is the order of magnitude of $ex(n, F)$ even determined. Kövári, Sós, and Turán [16] showed that for fixed r, s, where $2 \leq r \leq s$, $ex(n, K_{r,s}) = O(n^{2-1/r})$ as a function of n. Kollár, Rónyai, and Szabó [15] showed that for fixed r, s, where $r \geq 4$ and $s \geq r! + 1$, $ex(n, K_{r,s}) = \Omega(n^{2-1/r})$ as a function of n, thus establishing the order of magnitude for such $K_{r,s}$.

More generally, Alon, Krivelevich, and Sudakov [1] showed that if F is a bipartite graph in which vertices in one partite set all have degree at most r, then $ex(n, F) = O(n^{2-1/r})$. This verifies a special case of a long-standing conjecture of Erdős and

*Received by the editors December 23, 2010; accepted for publication (in revised form) May 21, 2012; published electronically September 4, 2012.
http://www.siam.org/journals/sidma/26-3/81925.html

†Department of Mathematics, Miami University, Oxford, OH 45056 (jiangt@muohio.edu, seiverrs@muohio.edu). The first author's research was partially supported by the National Security Agency under grant H98230-07-1-027.

Simonovits that $ex(n, F) = O(n^{2-1/r})$ for every r-degenerate bipartite graph, where F is *r-degenerate* if its vertices can be linearly ordered so that every vertex has a most r earlier neighbors. Equivalently, F is r-degenerate if $\max_{H\subseteq F} \delta(H) \leq r$. Towards proving the Erdős–Simonovits conjecture, Alon, Krivelevich, and Sudakov [1] showed that $ex(n, F) = O(n^{2-1/4r})$ for r-degenerate bipartite graphs F.

Erdős and Rényi [8] established a general lower bound on $ex(n, F)$ using random graphs, showing that $ex(n, F) = \Omega(n^{2-m/e})$ if F has m vertices and e edges. Using the deletion method, it is not too hard to improve their bound to $ex(n, F) = \Omega(n^{2-m/e+1/e})$ as described below. First, we introduce a definition.

Definition 1.1. *For a graph F, we define the* local-density $\gamma(F)$ *of F as* $\gamma(F) = \max_{H\subseteq F} \frac{e(H)}{n(H)-1}$.

Proposition 1.2. *Let F be a bipartite graph with m vertices and e edges. Then $ex(n, F) = \Omega(n^{2-\frac{m}{e}+\frac{1}{e}})$. More generally, we have $ex(n, F) = \Omega(n^{2-\frac{1}{\gamma}})$, where $\gamma = \gamma(F)$ is the local-density of F.*

The proof of Proposition 1.2 is a standard application of the deletion method. We postpone its proof to Appendix A. This improved lower bound is quite useful in some cases. For instance, when $F = C_{2k}$, the Erdős–Rényi bound gives only $ex(n, C_{2k}) = \Omega(n)$, but Proposition 1.2 gives $ex(n, C_{2k}) = \Omega(n^{1+1/2k})$, which was earlier obtained by Erdős [6]. The best known lower bound on $ex(n, C_{2k})$ is $\Omega(n^{1+2/(3k-3)})$, due to Lazebnik, Ustimenko, and Woldar [18] using an explicit construction. A long-standing conjecture of Erdős [7] states that $ex(n, C_{2k}) = \Omega(n^{1+1/k})$.

It is worth noting the following connection between Proposition 1.2 and the conjecture of Erdős and Simonovits on r-degenerate bipartite graphs. For an r-degenerate graph F, it is easy to see that $e(H) \leq r(n(H) - 1)$ holds for all $H \subseteq F$. Hence $\gamma(F) \leq r$. Furthermore, $K_{r,s}$, where $s \gg r$, shows that γ can be made arbitrarily close to r. This suggests that one possible motivation behind the Erdős–Simonovits conjecture was based on the local density of F. One might be tempted to make a stronger conjecture that $ex(n, F) = O(n^{2-1/\gamma})$ holds for all bipartite graphs F with local-density γ. This, however, is not true, as shown by the even cycles C_{2k}.

It is easy to see that a graph F with local-density γ is $\lfloor 2\gamma \rfloor$-degenerate. The Alon–Krivelevich–Sudakov bound and Proposition 1.2 thus yield the following result.

Proposition 1.3. *Let F be a bipartite graph. Let $\gamma = \max_{H\subseteq F} \frac{e(H)}{n(H)-1}$. Then there exist constants c_1, c_2 such that $c_1 n^{2-\frac{1}{\gamma}} \leq ex(n, F) \leq c_2 n^{2-\frac{1}{8\gamma}}$.*

In this paper, we study bipartite graphs F, where $ex(n, F)$ is small (close to being linear in n). The only graphs F with $ex(n, F) = O(n)$ are forests. For nonforests F, we may iteratively remove all vertices of degree 0 or 1, since doing so affects the Turán number by at most $O(n)$ and hence has no effect on the leading term in $ex(n, F)$. Thus, we may restrict our attention to bipartite graphs F with $\delta(F) \geq 2$. By Proposition 1.2, for $ex(n, F)$ to be close to being linear, the local density $\gamma(F)$ of F must be close to 1. Since $\delta(F) \geq 2$, this means that most vertices of F must have their degree equal to 2. Recall that the operation of *subdividing* an edge uv in a graph means replacing uv with uwv through a new vertex w of degree 2, and that a graph F is a *subdivision* of another graph H if F is obtained from H by subdividing edges of H. When a graph F has most of its vertices having degree 2, we may naturally view it as a subdivision of a much smaller graph H. Therefore, we will focus on studying $ex(n, F)$ when F is a subdivided graph.

Another motivation behind our study comes from the works by Jiang [14] and by Kostochka and Pyber [17] on topological minors (i.e., subdivisions). A well-known theorem of Mader [19] shows that for any graph H there is a constant c_H such that every n-vertex graph G with at least $c_H n$ edges contains a subdivision of H. However, there is no control on the order (number of vertices) of such a subdivision. It is easy to see by girth-type results that, to guarantee a subdivision of H of bounded order, $O(n)$ edges are not enough. It is thus natural to ask how many edges in G are sufficient to force a subdivision of H of bounded order. Erdős et al. [5] raised a question of this type by asking whether it is true that every n-vertex graph with at least $n^{1+\epsilon}$ edges contains a nonplanar subgraph of order at most $c(\epsilon)$, where $c(\epsilon)$ depends only on ϵ. This is equivalent to asking whether $n^{1+\epsilon}$ edges suffice to force a subdivision of K_5 or $K_{3,3}$ of order at most $c(\epsilon)$. Kostochka and Pyber [17] answered the question in the affirmative, proving a more general result (with the $t = 5$ case answering Erdős's question).

THEOREM 1.4 (Kostochka and Pyber [17]). *Let ϵ be a positive real such that $0 < \epsilon < 1$. Let n, t be positive integers. Every n-vertex graph G with at least $4^{t^2} n^{1+\epsilon}$ edges contains a subdivision of K_t on at most $\frac{7t^2 \ln t}{\epsilon}$ vertices.*

It is well known that for each $0 < \epsilon < 1$ and infinitely many n there are n-vertex graphs with $\Omega(n^{1+\epsilon})$ edges and girth at least $\frac{1}{\epsilon}$ (see [2, Chapter 3]). In such a graph, any subdivision of K_t, where $t \geq 3$, must contain at least $\Omega(\frac{t^2}{\epsilon})$ vertices. Kostochka and Pyber [17] asked whether their $O(\frac{t^2 \ln t}{\epsilon})$ bound on the order of a smallest subdivision of K_t in G can be improved to the optimal $O(\frac{t^2}{\epsilon})$. Recently, Jiang [14] answered Kostochka and Pyber's question in the affirmative, by showing that in an n-vertex graph G with $\Omega(n^{1+\epsilon})$ edges, there exists a subdivision of K_t in which each edge of K_t is subdivided $O(\frac{1}{\epsilon})$ times. (Note that such a subdivision has $O(\frac{t^2}{\epsilon})$ vertices.)

THEOREM 1.5 (Jiang [14]). *Let t be a positive integer and $0 < \epsilon < 1$ be a real. Let n be a sufficiently large positive integer as a function of t and ϵ. If G is an n-vertex graph with at least $n^{1+\epsilon}$ edges, then G contains a subdivision of K_t in which each edge of K_t is subdivided fewer than $\lfloor \frac{10}{\epsilon} \rfloor$ times.*

Remark 1.6. Due to the girth result mentioned above, the $\lfloor \frac{10}{\epsilon} \rfloor$-bound cannot be improved to be smaller than $\lfloor \frac{1}{3}(\lfloor \frac{1}{\epsilon} \rfloor - 1) \rfloor$.

Given positive integers p, t, let $K_t^{(\leq p)}$ denote the family of graphs obtained from K_t by subdividing each of its edges at most $p-1$ times. We may rephrase Theorem 1.5 as follows.

COROLLARY 1.7 (see [14]). *Let p, t be fixed positive integers, where $p \geq 2$ and $t \geq 3$. As a function of n, we have $ex(n, K_t^{(\leq p)}) = O(n^{1+\frac{10}{p}})$. Furthermore, $ex(n, K_t^{(\leq p)}) = \Omega(n^{1+\frac{1}{3p+1}})$.*

In this paper, we will establish analogous bounds on $ex(n, K_t^{(p)})$, where $K_t^{(p)}$ is the single graph obtained from K_t by subdividing each edge of K_t exactly $p-1$ times. Before we proceed, we would like to point out that there is a significant difference between bounding $ex(n, K_t^{(\leq p)})$ and bounding $ex(n, K_t^{(p)})$. For instance, it is very easy to show that every n-vertex graph with at least $cn^{1+1/k}$ edges contains a cycle of length at most $2k$. However, it is much harder to establish a similar result on guaranteeing a cycle of length exactly $2k$. Bondy and Simonovits [3] (and independently Erdős) showed that $ex(n, C_{2k}) = O(n^{1+1/k})$. Subsequent improvements on the leading coefficient were found by Verstraëte [23] and more recently by Pikhurko [20].

Our bounds on $ex(n, K_t^{(p)})$ follow from the more general bounds on $ex(n, F)$ for subdivided bipartite graphs F, described as below. Suppose a graph F is a obtained from another graph H by subdividing the edges of H. Then vertices of H form the set W of the *branch vertices* in F. For every pair $x, y \in W = V(H) \subseteq V(F)$ such that $xy \in E(H)$, there is a unique x, y-path in F that is internally disjoint from W.

We will call it the *strict* x,y*-path* in F and let $l_{x,y}$ denote its length. So, $l(x,y)-1$ is the number of times the edge xy in H is subdivided in forming F. We call $l_{x,y}$ the *stretch* of xy in F. Our main result is as follows.

THEOREM 1.8. *Let F be a subdivision of another graph H. For each edge $xy \in E(H)$, let $l_{x,y}$ denote the stretch of xy in F. Suppose that $l_{x,y}$ is even for all x,y, where $xy \in E(H)$, and that* $\min\{l_{x,y} : xy \in E(H)\} = 2m$. *Then $ex(n,F) = O(n^{1+\frac{8}{m}})$.*

Theorem 1.8 and Proposition 1.2 immediately yield the following result.

COROLLARY 1.9. *Let p,t be positive integers, where p is even. As a function of n, we have $ex(n,K_t^{(p)}) = O(n^{1+\frac{16}{p}})$. Also, $ex(n,K_t^{(p)}) = \Omega(n^{1+\frac{1-1/t}{p}})$.*

Note that it is natural to restrict our attention to the even p case since when p is odd, $K_t^{(p)}$ is nonbipartite and $ex(n,K_t^{(p)}) = \Omega(n^2)$. For convenience, we will prove the following equivalent version of Theorem 1.8.

THEOREM 1.10. *Let ϵ be a real where $0 < \epsilon < 1$. Let F be a subdivision of another graph H. For every edge $xy \in E(H)$, let $l_{x,y}$ denote the stretch of xy in F. Suppose that $l_{x,y}$ is even and $l_{x,y} \geq 2\lfloor\frac{8}{\epsilon}\rfloor$ for all $x,y \in W$, where $xy \in E(H)$. Then for some positive constant c_F, depending on F, every n-vertex graph G with at least $c_F n^{1+\epsilon}$ edges contains F as a subgraph.*

We would like to mention that in establishing the $O(n^{1+\frac{8}{m}})$ bound in Theorem 1.8 and hence the $O(n^{1+\frac{16}{p}})$ bound on $ex(n,K_t^{(p)})$, our main goal was to establish an $O(n^{1+\frac{c}{m}})$ bound for an absolute constant c, and thus we did not spend too much effort optimizing the constant c. Indeed, at first glance, it is not clear at all whether one should expect a $O(n^{1+\frac{c}{p}})$ bound on $ex(n,K_t^{(p)})$, where c is an absolute constant independent of t. It would be a very interesting problem to substantially reduce this constant c. By Corollary 1.9, c cannot be reduced to be smaller than 1.

For the rest of the paper, we prove Theorem 1.10. Our approach builds upon that of Jiang [14], which in turn incorporated ideas in [10] and [17]. Several crucial new ideas will be used to overcome the greater technical challenges (compared to those in [14]). The rest of the paper is organized as follows. In section 2, we prove some preliminary lemmas. In section 3, the most technical section, we solve the main case. Then in section 4, we put all the pieces together.

2. Preliminaries. As mentioned in the introduction, Alon, Krivelevich, and Sudakov [1] proved the following result.

THEOREM 2.1 (see [1]). *Let F be a bipartite graph with maximum degree r on one side. Then there exists a constant $c_F > 0$, depending on F, such that $ex(n,F) \leq c_F n^{2-\frac{1}{r}}$.*

Theorem 2.1 is also implicit in Füredi [12]. From the proof of Theorem 2.2 in [1], one can check that $c_F \leq n(F)$ when $r = 2$. Note that if F is obtained from another graph H by subdividing each edge an odd number of times, then the branch vertices all lie in the same partite set of F, and vertices in the other partite set have degree at most 2. We have the following.

COROLLARY 2.2. *Let F be a bipartite graph with v vertices such that vertices in one partite set all have degree at most 2; then $ex(n,F) \leq vn^{\frac{3}{2}}$. In particular, if F is obtained from another graph H by subdividing each edge an odd number of times, then $ex(n,F) \leq vn^{\frac{3}{2}}$.*

By Corollary 2.2, Theorem 1.10 holds whenever $\frac{1}{2} < \epsilon < 1$. Thus, we henceforth restrict our attention to $\epsilon \leq \frac{1}{2}$. The next two lemmas were used in [14]. We include their proofs here.

LEMMA 2.3. *Let a,m,q be positive integers. Let $A_1,\ldots,A_m$ be a collection of* *sets of size a. Suppose each element of $A = \bigcup_i A_i$ lies in at most q different A_i's. Then for each $i \in [m]$ there exists $B_i \subseteq A_i$ of size $\lfloor a/q \rfloor$ such that $B_1, \ldots, B_m$ are pairwise disjoint.*

Proof. Let $p = \lfloor a/q \rfloor$. Create a bipartite graph H with a bipartition (X, A), where $|X| = mp$ as follows. Label the vertices of X by $x_1^1, \ldots, x_1^p, x_2^1, \ldots, x_2^p, \ldots, x_m^1, \ldots, x_m^p$. For each $i \in [m]$ and $y \in A$, if $y \in A_i$, then we add edges between y and $x_i^1, \ldots, x_i^p$. By our construction, each vertex in X has degree a. Also, since each $y \in A$ lies in at most q different A_i's, each vertex in A has degree at most $pq \le a$ in H. By Hall's theorem, H has a matching M saturating all of X. For each $i \in [m]$, the elements of A that $x_i^1, \ldots, x_i^p$ are matched to by M are elements of A_i. Hence, we obtain disjoint $B_1, \ldots, B_m$ each of size $p = \lfloor a/q \rfloor$, with $B_i \subseteq A_i$ for each $i \in [m]$. ∎

The *depth* of a vertex x in a rooted tree T is the distance between the root and x. The depth of the tree is the maximum depth of a vertex in T. If T' is a tree rooted at u in which for every pair of leaves x and x' the unique x, x'-path goes through u, then we call T' a *spider* with center u. The paths from u to the leaves are called the *legs* of the spider.

LEMMA 2.4. *Let p, m be positive integers. Let T be a tree rooted at u of depth m. Let W be a set of p^m vertices in $V(T) - u$ at depth m. Then there exists a subset $W' \subseteq W$ of size p and a vertex z at some depth $j \le m-1$ such that the union of the paths in T from z to vertices in W' forms a spider T' whose set of leaves is W'. Furthermore, each leg of T' has length $m - j$.*

Proof. We use induction on the depth m. The claim holds trivially when $m = 1$. For the induction step, assume that $m > 1$ and that the claim holds for trees of depth at most $m - 1$. For each child x of u, let T_x denote the subtree of T under x. If for at least p different children x of u, $V(T_x) \cap W \neq \emptyset$, then let W' be a set of p vertices of W, each from a different T_x. The union of the paths from u to these vertices forms a spider T' with center u whose set of leaves is W'. Each leg of T' has length t.

Next, we may assume that $|V(T_x) \cap W| \neq \emptyset$ for fewer than p children x of u. Then for some child z of u, $|V(T_z) \cap W - z| \ge |W|/(p-1) - 1 \ge p^{m-1}$. Since T_z is a rooted tree of depth $m - 1$ and $W \cap V(T_z)$ is a set of vertices at depth $m - 1$ in T_z, We can apply induction hypothesis to find the desired W' and T'. ∎

The following lemma is folklore and can be easily proved using induction on k.

LEMMA 2.5. *Let k be a positive integer, G a graph with minimum degree at least k, and T a rooted tree with k edges. Let x be any vertex in G. There exists a copy of T in G with x being the image of the root.*

In the next section, we will first establish some useful results for "almost regular" graphs, namely graphs in which the maximum degree is within a constant factor of the minimum degree. In order to extend such results to general graphs, we need a variant of the following lemma of Erdős and Simonovits [9].

LEMMA 2.6 (see [9]). *Let ϵ be a real satisfying $0 < \epsilon < 1$. Let n be a positive integer that is sufficiently large as a function of ϵ. Let G be an n-vertex graph with $e(G) \ge n^{1+\epsilon}$. Then G contains a subgraph G' on $m \ge n^{\epsilon\frac{1-\epsilon}{1+\epsilon}}$ vertices such that $e(G') \ge \frac{2}{5} m^{1+\epsilon}$ and $\frac{\Delta(G')}{\delta(G')} \le c_\epsilon$, where $c_\epsilon = 10 \cdot 2^{\frac{1}{\epsilon^2}+1}$.*

By slightly modifying Erdős and Simonovits's proof, we obtain the following variant.

PROPOSITION 2.7. *Let ϵ, c be positive reals, where $\epsilon < 1$ and $c \ge 1$. Let n be a positive integer that is sufficiently large as a function of ϵ. Let G be an n-vertex graph with $e(G) \ge cn^{1+\epsilon}$. Then G contains a subgraph G' on $m \ge n^{\frac{\epsilon}{2}\frac{1-\epsilon}{1+\epsilon}}$ vertices such that $e(G') \ge \frac{2c}{5} m^{1+\epsilon}$ and $\frac{\Delta(G')}{\delta(G')} \le c_\epsilon$, where $c_\epsilon = 20 \cdot 2^{\frac{1}{\epsilon^2}+1}$.*

For completeness, we give a proof of Proposition 2.7 in Appendix B.

3. Dense graphs with no dense compact subgraphs. For convenience, in this section, we will assume our host graph G to be bipartite. We lose little generality in doing so, since G contains a bipartite subgraph with at least half of the edges. To prove Theorem 1.10, we first prove in this section that if a (bipartite) graph is quite dense itself but contains no large dense subgraph of small radius, then we can find a copy of the desired subdivided graph F. The following notion will be used frequently in our proofs.

DEFINITION 3.1. *Let c, ϵ be positive reals, where $\epsilon < 1$. A graph G is called (c, ϵ)-dense if $e(G) \geq c[n(G)]^{1+\epsilon}$.*

Given two disjoint subsets $A, B \subseteq V(G)$, we use $e_G(A, B)$ to denote the number of edges in G with one end in A and the other end in B. We omit the subscript G when the context is clear.

LEMMA 3.2. *Let c, λ, ϵ be positive reals with $\epsilon \leq \frac{1}{2}$. Let R be a positive integer. Let n be a positive integer. Let G be an n-vertex graph that contains no $(\frac{c}{2^7}, \epsilon)$-dense subgraph of radius at most R.*

1. *Let H be a subgraph of G of radius at most $R-1$ and order $n(H) \leq \lambda n^a$, where $a < 1$. Let W be a set of neighbors of $V(H)$ outside $V(H)$ such that $e(W, V(H)) \geq \frac{c}{16} n^{a+\epsilon}$. Then $|W| \geq 4n^{a+\frac{\epsilon}{1+\epsilon}(1-a)} - \lambda n^a$. In particular, if $\lambda \leq 2$, then $|W| \geq 2n^{a+\frac{\epsilon}{1+\epsilon}(1-a)}$.*
2. *Suppose further that $\delta(G) \geq cn^\epsilon$. If H is a subgraph of radius at most $R-1$ and W is the set of vertices outside H that have neighbors in $V(H)$, then $|W| \geq 9n^{\frac{\epsilon}{1+\epsilon}} \cdot [n(H)]^{\frac{1}{1+\epsilon}}$.*

Proof. 1. Let F denote the subgraph of G induced by $V(H) \cup W$. Since H has radius at most $R-1$, F has radius at most R. By our assumption, F is not $(\frac{c}{2^7}, \epsilon)$-dense. Suppose $n(F) = n^b$. We have

$$\frac{c}{16} n^{a+\epsilon} \leq e(F) \leq \frac{c}{2^7} (n^b)^{1+\epsilon}. \tag{1}$$

From this, we get

$$n^b \geq 8^{\frac{1}{1+\epsilon}} n^{\frac{a+\epsilon}{1+\epsilon}} \geq 4n^{a+\frac{\epsilon}{1+\epsilon}(1-a)}.$$

Hence, $|W| = n(F) - n(H) \geq 4n^{a+\frac{\epsilon}{1+\epsilon}(1-a)} - \lambda n^a$. If $\lambda \leq 2$, then $|W| \geq 2n^{a+\frac{\epsilon}{1+\epsilon}(1-a)}$.

2. Let H' be the subgraph of G induced by $V(H) \cup W$. Since $\delta(G) \geq cn^\epsilon$, there are at least $\frac{1}{2} n(H) \cdot cn^\epsilon$ edges of G that are incident to $V(H)$, all of which lie in H'. Since H' has radius at most R, H' is not $(\frac{c}{2^7}, \epsilon)$-dense. Hence we have

$$\frac{1}{2} cn^\epsilon \cdot n(H) \leq e(H') \leq \frac{c}{2^7} [n(H')]^{1+\epsilon}.$$

Solving the inequalities for $n(H')$ and using $2^{\frac{6}{1+\epsilon}} \geq 10$, we get $n(H') \geq 10n^{\frac{\epsilon}{1+\epsilon}} \cdot [n(H)]^{\frac{1}{1+\epsilon}}$. Hence, $|W| = n(H') - n(H) \geq 9n^{\frac{\epsilon}{1+\epsilon}} \cdot [n(H)]^{\frac{1}{1+\epsilon}}$. ∎

The following lemma deals with a recurrence that will be frequently used in our proofs.

LEMMA 3.3. *Let ϵ be a real with $0 < \epsilon \leq 0.5$. The recurrence relation $a_{i+1} = a_i + \frac{\epsilon}{1+\epsilon}(1-a_i)$, $i \geq 1$, has the solution $a_i = 1 - \frac{1-a_1}{(1+\epsilon)^{i-1}}$. Thus, $a_i \geq 1-(1-a_1)e^{-0.8\epsilon(i-1)}$.*

Proof. The first part can be directly verified by induction. For $0 < \epsilon \leq 0.5$, we have $\ln(1+\epsilon) \geq 0.8\epsilon$ and hence $1+\epsilon \geq e^{0.8\epsilon}$. Thus, we conclude $a_i = 1 - \frac{1-a_1}{(1+\epsilon)^{i-1}} \geq 1 - \frac{1-a_1}{e^{0.8\epsilon(i-1)}}$. ∎

The next structural lemma is crucial to our arguments in this section.

LEMMA 3.4. *Let c, ϵ, b be positive reals, where $\epsilon \leq \frac{1}{2}$ and $\epsilon \leq b \leq 0.9$. Let γ_ϵ be a positive real depending on ϵ. Let k, r, R be positive integers, where $r \leq R$. Let G be an n-vertex bipartite graph with $\delta(G) \geq cn^\epsilon$ that contains no $(\frac{c}{2^7}, \epsilon)$-dense subgraph of radius at most R, where n is sufficiently large as a function of ϵ and k. Let (X, Y) be a bipartition of G. Let F be a subgraph of G of radius at most r and $S \subseteq X \cap V(F)$ such that $|S| = n^b$ and $n(F) \leq \gamma_\epsilon |S|$. Then there exist disjoint sets $L_1, \ldots, L_{R-r+1}$, where $L_1 = S$, outside $V(F) - S$ satisfying the following:*

1. *Each L_i is called a* level *and will be designated as* strong *or* weak. *Set L_1 will be strong.*
2. *Each L_i is partitioned into some d_i subsets L_i^j called* sectors *of equal size. If L_i is a strong level, then each sector consists of only a single vertex. For $i \geq 2$, if L_i is strong, then $d_i \geq d_{i-1}$, and if L_i is weak, then $\frac{1}{8} d_{i-1} \leq d_i \leq d_{i-1}$.*
3. *Let $i \geq 2$. If L_i is a strong level, then each vertex in L_i has neighbors in at least k sectors of L_{i-1}. If L_i is a weak level, then there is an injection f from the collection of sectors of L_i into the collection of sectors of L_{i-1} such that if L_i^j is a sector of L_i, then each vertex in L_i^j has at least one neighbor in $f(L_i^j)$. We call $f(L_i^j)$ the* parent sector *of L_i^j in L_{i-1}. Furthermore, $|L_i^j| \geq 2|f(L_i^j)|$.*
4. *Let $i \geq 2$. If $x_1, \ldots, x_m$, where $m \leq k$, all belong to different sectors of L_i, then there exist $y_1, \ldots, y_m$ belonging to different sectors of L_{i-1} such that $x_1 y_1, \ldots, x_m y_m \in E(G)$.*
5. *For each i, suppose $|L_i| = n^{a_i}$ and $d_i = n^{\lambda_i}$. Then $|L_{i+1}| \geq 2|L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i)}$. Thus, $a_{i+1} \geq a_i + \frac{\epsilon}{1+\epsilon}(1-a_i)$. Further, if L_{i+1} is a weak level, then $|L_{i+1}| \geq |L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.99\lambda_i)}$.*

Furthermore, if $0.2 \leq b \leq 0.9$ and $R - r \geq l(b, \epsilon) = \lceil \frac{\ln((1-b)/0.19)}{0.8\epsilon} \rceil + \lceil \frac{0.2}{b}(\frac{1}{\epsilon}+1) \rceil$, then for some s, $2 \leq s \leq l(b, \epsilon) + 1$, L_s is a strong level and $|L_s|, |L_{s-1}| \geq n^{0.81}$.

Proof. Let $L_0 = V(F) - S$. We iteratively define the L_i's that satisfy conditions 1–5. Note that condition 4 follows immediately from condition 3. Thus, we need to verify only conditions 1–3 and 5 in our constructions. As we construct L_i, we will also let H_i denote the subgraph of G induced by $L_0 \cup L_1 \cup \cdots \cup L_i$.

To start, let $L_1 = S$. Designate L_1 as a strong level, and designate each vertex in L_1 as a sector on its own. Fix $2 \leq i \leq R - r$, and suppose for all $j \leq i$ that we have defined L_j that satisfy 1–5. We describe how to construct L_{i+1}. The numerical details are slightly different for $i = 1$ versus for $i \geq 2$.

For $i = 1$, we have $n(H_1) = |L_0| + |L_1| \leq (\gamma_\epsilon + 1)|L_1|$ by our assumption. For $i \geq 2$, we note that, by condition 5, $|L_2| \geq 2|L_1| n^{\frac{\epsilon}{1+\epsilon}(1-a_1)} \geq 2|L_1| n^{\frac{0.1\epsilon}{1+\epsilon}} \geq 2(\gamma_\epsilon + 1)|L_1| \geq 2(|L_0| + |L_1|)$ for large n, and for each $3 \leq j \leq i$, $|L_j| \geq 2|L_{j-1}|$. So, $n(H_i) = |L_0| + |L_1| + \cdots + |L_i| \leq |L_i|(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \leq 2|L_i|$.

By 3 and 4, for each i, each vertex in L_i has at least one neighbor in L_{i-1}. Since F has radius at most r, H_i has radius at most $r + i - 1 \leq R - 1$.

Let d denote the minimum degree of G. We have $d \geq cn^\epsilon$. We say that a vertex x in L_i is *bad* if at least $\frac{d}{2}$ of its neighbors lie inside H_i. Otherwise we say it is *good*. Note that a good vertex has at least $\frac{d}{2}$ neighbors outside H_i. Let B_i denote the set of bad vertices in L_i, and C_i the set of good vertices in L_i. Consider first the case when $i = 1$. Suppose $|B_1| \geq \frac{1}{4}|L_1|$. Then $|B_1| \geq \frac{1}{4(\gamma_\epsilon+1)} n(H_1)$ and $e(H_1) \geq \frac{1}{2} \frac{1}{4(\gamma_\epsilon+1)} n(H_1) \cdot \frac{cn^\epsilon}{2} \geq c[n(H_1)]^{1+\epsilon}$, for large n (noting that $n(H_1) = O(n^b)$ and $b \leq 0.9$). This contradicts G having no $(c/2^7, \epsilon)$-dense subgraph of radius at most R. So $|B_1| \leq \frac{1}{4}|L_1|$ and $|C_1| \geq \frac{3}{4}|L_1|$. Next, suppose $i \geq 2$. If $|B_i| > \frac{1}{4}|L_i|$, then

$|B_i| \geq \frac{1}{8}n(H_i)$ and $e(H_i) \geq \frac{1}{2}|B_i| \cdot \frac{d}{2} \geq \frac{1}{16}n(H_i) \cdot \frac{c}{2}n^\epsilon \geq \frac{c}{32}[n(H_i)]^{1+\epsilon}$, contradicting G containing no $(\frac{c}{2^7}, \epsilon)$-subgraph with radius at most R. Hence $|B_i| \leq \frac{1}{4}|L_i|$ and $|C_i| \geq \frac{3}{4}|L_i|$.

Let W denote the set of vertices outside H_i that have neighbors in L_i. We now analyze the vertices in $L_i \cup W$ and the edges between L_i and W. We say a vertex y in W is *heavy* if it has neighbors in at least k different sectors L_i^j of L_i. Otherwise we say y is *light*. Let W^+ denote the set of heavy vertices in W, and W^- the set of light vertices in W.

Recall that each vertex x in C_i sends at least $\frac{d}{2}$ edges to W. If at least $\frac{d}{4}$ of these edges go to W^+, we say that x is *heavy-leaning*. Otherwise at least $\frac{d}{4}$ of these edges go to W^- and we say that x is *light-leaning*. Let C_i^+ denote the set of heavy-leaning vertices in C_i and C_i^- the set of light-leaning vertices in C_i. We consider two cases. Recall that $|L_i| = n^{a_i}$.

Case 1. $|C_i^+| \geq |C_i|/2$. In this case, we have $|C_i^+| \geq \frac{3}{8}|L_i|$. We have $e(V(H_i), W^+) \geq e(C_i^+, W^+) \geq \frac{3}{8}|L_i| \cdot \frac{d}{4} \geq \frac{3c}{32}n^{a_i+\epsilon}$. Suppose first that $i = 1$. Then $n(H_1) \leq (\gamma_\epsilon + 1)|L_1| = (\gamma_\epsilon + 1)n^{a_1}$ and H_1 has radius at most $R - 1$. By Lemma 3.2, $|W^+| \geq 4n^{a_1+\frac{\epsilon}{1+\epsilon}(1-a_1)} - (\gamma_\epsilon + 1)n^{a_1} \geq 2n^{a_1+\frac{\epsilon}{1+\epsilon}(1-a_1)} = 2|L_1| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_1)}$ for large n. Next, suppose $i \geq 2$. Then $n(H_i) \leq 2|L_i| = 2n^{a_i}$, and H_i has radius at most $R - 1$. By Lemma 3.2, $|W^+| \geq 2n^{a_i+\frac{\epsilon}{1+\epsilon}(1-a_i)} = 2|L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i)}$. Let $L_{i+1} = W^+$, and designate L_{i+1} as a strong level. Let each vertex be a sector by itself. It is straightforward to see that L_{i+1} satisfies 1–5.

Case 2. $|C_i^-| \geq |C_i|/2$. This case is more involved. Our plan is to define L_{i+1} inside W^-. Roughly speaking, sectors of L_{i+1} will be defined using neighborhoods of certain sectors of L_i in W^-. We will use Lemma 2.3 to handle the overlaps between these neighborhoods.

First, we decide which sectors of L_i to use. We know $|C_i^-| \geq \frac{|C_i|}{2} \geq \frac{3}{8}|L_i|$. Let $J = \{j : |L_i^j \cap C_i^-| \geq \frac{|L_i^j|}{4}\}$. Since all the d_i sectors L_j^i of L_i have the same size and $\sum_j |L_i^j \cap C_i^-| \geq \frac{3}{8}|L_i|$, we have $|J| \geq \frac{d_i}{8}$. We will use the neighborhoods of L_i^j in W^- for those $j \in J$.

For each $j \in J$, let $W_i^j = N(L_i^j) \cap W^-$. Suppose $|L_i^j| = n^{a_{i,j}}$. First we argue that there is a subgraph T_i^j of order at most $2n^{a_{i,j}}$ and radius at most $R - 1$ containing L_i^j. If L_i is a strong level, then this is trivial, since L_i^j consists of a single vertex and we can let T_i^j be that vertex. If L_i is a weak level, then $i \geq 2$, and by 3, each vertex in L_i^j has some neighbor in its parent sector $f(L_i^j)$ in L_{i-1}. Furthermore, $|f(L_i^j)| \leq \frac{1}{2}|L_i^j|$. We can continue backtracking, moving up the levels, until we hit a strong level (where a sector is just a single vertex). This gives us a subgraph T_i^j of order at most $|L_i^j|(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) \leq 2n^{a_{i,j}}$ that has radius at most $i - 1 \leq R - 1$. Now, since $j \in J$, we have $e(L_i^j, W_i^j) = e(L_i^j, W^-) \geq |L_i^j \cap C_i^-| \cdot \frac{d}{4} \geq \frac{1}{4}|L_i^j| \cdot \frac{d}{4} \geq \frac{c}{16}n^{a_{i,j}+\epsilon}$. By Lemma 3.2, $|W_i^j| \geq 2n^{a_{i,j}+\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$. Let Y_i^j be a subset of W_i^j of size $2n^{a_{i,j}+\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}$. Since $|L_i^j| = 2n^{a_{i,j}}$ is the same for all j, $|Y_i^j|$ is the same for all $j \in J$.

Note that $\bigcup_{i\in J} Y_i^j \subseteq W^-$ and by the definition of W^- each element of W^- lies in at most k different Y_i^j's. By Lemma 2.3, for each $j \in J$, there exists a subset $A_i^j \subseteq Y_i^j$ of size $|Y_i^j|/k$ such that the A_i^j's so obtained are pairwise disjoint. We let $L_{i+1} = \bigcup_{j\in J} A_i^j$. Let $d_{i+1} = |J|$. The A_i^j's for $j \in J$ form a partition of L_{i+1} into d_{i+1} many subsets of equal size. We define them to be the sectors of L_{i+1} and designate L_{i+1} as a weak level. It remains to verify that 2, 3, and 5 hold for L_{i+1}. We have

$d_{i+1} = |J| \geq \frac{1}{8}d_i$, so 2 holds. For 3, for each $j \in J$ we let $f(A_i^j) = L_i^j$. Finally, we rename the A_i^j's, $j \in J$, as $L_{i+1}^1, \dots, L_{i+1}^{d_{i+1}}$.

Recall that $|L_i| = n^{a_i}$ and $d_i = n^{\lambda_i}$. For each $j \in J$, $|L_j^i| = |L_i|/d^i = n^{a_i-\lambda_i}$. Thus, $a_{i,j} = a_i - \lambda_i$. By 2, $d_i \geq \frac{d_1}{8^{i-1}} = \frac{n^b}{8^{i-1}} \geq n^{0.99b}$ for large n. So, $\lambda_i \geq 0.99b$. Since $|J| \geq \frac{d_i}{8}$, and for each $j \in J$, $|A_j^i| = 2n^{a_{i,j}+\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}/k = 2|L_j^i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})}/k$, we have $|L_{i+1}| \geq \frac{d_i}{8k} \cdot 2|L_j^i| n^{\frac{\epsilon}{1+\epsilon}(1-a_{i,j})} = \frac{1}{4k}|L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+\lambda_i)} \geq |L_i| \cdot n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.99\lambda_i)}$ for large n. Thus, 5 holds for L_{i+1}.

We have now constructed the sets $L_1, L_2, \dots, L_{R-r+1}$ that satisfy conditions 1–5. Note that, by 5, $|L_i|$ is nondecreasing in i. It remains to prove the last statement of the lemma. So, suppose $0.2 \leq b \leq 0.9$ and $R - r \geq l(b, \epsilon) = \lceil \frac{\ln((1-b)/0.19)}{0.8\epsilon} \rceil + \lceil \frac{0.2}{b}(\frac{1}{\epsilon}+1) \rceil$. We first find an i for which $a_i \geq 0.81$. By Lemma 3.3, $a_i \geq 1 - (1-a_1)e^{-0.8\epsilon(i-1)} = 1 - (1-b)e^{-0.8\epsilon(i-1)}$. So it suffices if $1 - (1-b)e^{-0.8\epsilon(i-1)} \geq 0.81$. Solving for i, we get $i \geq \frac{\ln((1-b)/0.19)}{0.8\epsilon} + 1$. Let $t = \min\{i : a_i \geq 0.81\}$. Then $t \leq \lceil \frac{\ln((1-b)/0.19)}{0.8\epsilon} \rceil + 1$ $= l(b,\epsilon) + 1 - \lceil \frac{0.2}{b}(\frac{1}{\epsilon}+1) \rceil$.

If L_s is a strong level for some $s \in I = \{t+1, \dots, l(b,\epsilon)+1\}$, then $2 \leq s \leq l(b,\epsilon)+1$ and $|L_s| \geq |L_{s-1}| \geq |L_t| \geq n^{0.81}$, and we are do done. So suppose for each $i \in I$ that L_i is a weak level. By 5, $\frac{|L_i|}{|L_{i-1}|} \geq n^{\frac{\epsilon}{1+\epsilon}(1-a_i+0.99\lambda_i)} > n^{\frac{\epsilon}{1+\epsilon}(0.98b)}$ (recalling that $\lambda_i \geq 0.99b$). But then, since $l(b,\epsilon) + 1 - t \geq \lceil \frac{0.2}{b}(\frac{1}{\epsilon}+1) \rceil$, we have $|L_{l(b,\epsilon)+1}| > |L_t| \cdot n^{\frac{\epsilon}{1+\epsilon}(0.98b)\cdot\frac{0.2}{b}(\frac{1}{\epsilon}+1)} > n^{0.81} \cdot n^{0.19} = n$, a contradiction. ∎

DEFINITION 3.5. *Let p, q_1, q_2 be positive integers, where $q_1 \leq q_2$ and q_1, q_2 have the same parity. We define a* (p, q_1, q_2)-path system *between two vertices x and y to be a collection of $p(\frac{q_2-q_1}{2}+1)$ internally disjoint x,y-paths in G such that, for each integer j between q_1 and q_2 having the same parity as them, p of these paths have length j. If a (p, q_1, q_2)-path system exists between x and y, we say that x, y are (p, q_1, q_2)-linked. We also say that x is (p, q_1, q_2)-linked to y and vice versa.*

Let $S \subseteq G$, where $y \notin S$. We say that S is super-(p, q_1, q_2)-linked *to y if for every vertex $x \in S$ there is a (p, q_1, q_2)-path system between x and y that is internally disjoint from S.*

LEMMA 3.6. *Let c, ϵ, D be positive reals, where $\epsilon \leq 0.5$ and $c \geq 2$. Let m be a positive integer. Let $R = \lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil + 1$. Let n be a positive integer that is sufficiently large as a function of ϵ and m . Let G be an n-vertex bipartite graph satisfying* (a) $\delta(G) \geq cn^\epsilon$, (b) $\Delta(G) \leq Dn^\epsilon$, *and* (c) *$G$ contains no $(\frac{c}{2^7}, \epsilon)$-dense subgraph of radius at most R. Then there exist at least $n^{0.74}$ vertices (called* supervertices*) z satisfying that for some positive integer $q(z) \leq R-1$ depending on z there is a set U_z of size at least $n^{0.6}$ that is super-$(m, q(z), q(z))$-linked to z.*

Proof. Let w be any vertex in G. For each $i = 0, \dots, R$, let n_i denote the number of vertices at distance at most i from w in G. By Lemma 3.2, for each $i = 1, \dots, R$, $n_i \geq 10n^{\frac{\epsilon}{1+\epsilon}} n_{i-1}^{\frac{1}{1+\epsilon}}$. Since $n_1 \geq cn^\epsilon \geq 2n^\epsilon$, by induction, one can easily show that $n_i \geq 2n^{1-\frac{1-\epsilon}{(1+\epsilon)^{i-1}}}$ for each $1 \leq i \leq R$. Let $h = \min\{i : n_i \geq 2n^{0.2}\}$. If $\epsilon \geq 0.2$, then clearly $h = 1 \leq \lceil \frac{1}{4\epsilon} \rceil$. Suppose $\epsilon < 0.2$. Then $1 + \epsilon > e^{0.9\epsilon}$. For $i = \lceil \frac{1}{4\epsilon} \rceil$, we have $1 - \frac{1-\epsilon}{(1+\epsilon)^{i-1}} \geq 1 - \frac{1}{(1+\epsilon)^i} > 1 - \frac{1}{e^{0.9\epsilon i}} \geq 1 - \frac{1}{e^{0.9\epsilon/4\epsilon}} \geq 0.2$. So $h \leq \lceil \frac{1}{4\epsilon} \rceil$. Since $\Delta(G) \leq Dn^\epsilon$, $n_h \leq n_{h-1} \cdot Dn^\epsilon < 2n^{0.2} \cdot Dn^\epsilon < n^{0.9}$ for large n.

Let T_w denote the tree rooted at w obtained by applying the breadth-first search from w in G for h steps. Then T has radius $h \leq \lceil \frac{1}{4\epsilon} \rceil$. Let S be the set of all the vertices in T_w that are at distance h from w. Then S is a set of leaves, and $|S| = n^b$ for some b, $0.2 \leq b \leq 0.9$. Since G is bipartite, S lies inside one partite set of a bipartition of G. Also, $|S| = n_h - n_{h-1}$. By our discussion above, $n_h \geq 10n_{h-1}$. So, $n(F) \leq \frac{11}{10}|S|$.

Let $F = T_w$, and let $k = m^h$. By Lemma 3.4, there exist disjoint sets $L_1, \dots, L_{R-h+1}$ that satisfy conditions 1–5 of the lemma. Also, by the last statement of Lemma 3.4, for some $s \leq l(b,\epsilon)+1$, L_s is a strong level and $|L_s|, |L_{s-1}| \geq n^{0.81}$. Note that $l(b,\epsilon) \leq l(0.2,\epsilon) = \lceil \frac{\ln(0.8/.19)}{0.8\epsilon} \rceil + \lceil \frac{0.2}{0.2}(\frac{1}{\epsilon}+1) \rceil \leq \lceil \frac{1.8}{\epsilon} \rceil + \lceil \frac{1}{\epsilon} \rceil + 1 \leq \frac{2.8}{\epsilon} + 3 \leq \frac{4.3}{\epsilon}$, where the last inequality uses $\epsilon \leq 0.5$. So, $s \leq \lfloor \frac{4.3}{\epsilon} \rfloor + 1$.

Let x be any vertex in L_s. Since L_s is strong, x has neighbors in at least k different sectors of L_{s-1}. By repeatedly applying condition 4 of Lemma 3.4 for $i = s-1, s-2, \dots, 2$, we can build k paths of length $s-1$ from x to L_1 through $L_1 \cup \cdots \cup L_{s-1}$ such that every two of them share only x in common. Let $P_1, \dots, P_k$ denote such paths, and let $w_1, \dots, w_k$ denote their endpoints in L_1, respectively. Let $W = \{w_1, \dots, w_k\}$. Since $k = m^h$, by Lemma 2.4, there exists z in T_w at some depth $depth(z) \leq h-1$ and a subset $W' \subseteq W$ of size m such that the union of the paths in T_w from z to W' forms a spider of m legs each of length $h - depth(z)$. This spider together with the P_i's yields p internally disjoint z,x-paths of length $q(z) = (s-1) + h - depth(z) \leq \lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil$. Also, by our construction, these paths are internally disjoint from L_s. By the argument above, we can define a mapping g from L_s into the set of vertices at depth at most $h-1$ in T_w such that for each vertex $x \in L_s$, with $z = g(x)$, there exists an $(m, q(z), q(z))$-path system between x and z that is internally disjoint from L_s. By our choice of h, there are fewer than $2n^{0.2}$ vertices at depth $h-1$ or less in T_w. So, by the pigeonhole principle, for some z in T_w at depth $depth(z) \leq h-1$, the set $U_z = \{x : x \in L_s, g(x) = z\}$ has size at least $|L_s|/2n^{0.2} \geq n^{0.81}/2n^{0.2} \geq n^{0.6}$ for large n. The set U_z is super-(m, q_1, q_2)-linked to z.

We have shown that for each vertex w, T_w contains a supervertex z. On the other hand, for each supervertex z, for z to lie in some T_w, w must be within distance $h - 1 \leq \lceil \frac{1}{4\epsilon} \rceil - 1$ from z, and there are at most $\sum_{i=0}^{\lceil \frac{1}{4\epsilon} \rceil - 1} (Dn^\epsilon)^i < n^{0.26}$ such w (for large n). Hence the number of supervertices is at least $n/n^{0.26} = n^{0.74}$. ∎

LEMMA 3.7. *Let m, p, q be positive integers, where $m \geq pq$. Let G be a graph. Let $x \in V(G)$ and $S \subseteq V(G)$, where $x \notin S$. Suppose S is super-(m,q,q)-linked to x. Then for any p vertices $y_1, \dots, y_p \in S$ there exist paths $Q_1, \dots, Q_p$ of length q such that, for each $i \in [p]$, Q_i connects x and y_i and such that every two of these paths share only x in common.*

Proof. By our assumption, for each i there exist m internally disjoint x, y_i-paths of length q that are internally disjoint from S. We build the paths $Q_1, \dots, Q_p$ one by one. To start, let Q_1 be any x, y_1-path of length q that is internally disjoint from S. In general, let $1 \leq i \leq p-1$, and suppose that $Q_1, \dots, Q_i$ have been defined. Let $U = V(Q_1 \cup \cdots \cup Q_i)$. Then $|U| < iq < m$ and $U \cap S = \{y_1, \dots, y_i\}$. Since there are m internally disjoint x, y_{i+1}-paths of length q that are internally disjoint from S, we can pick one that doesn't go through any vertex in $U - x$ and define it to be Q_{i+1}. We can continue till $Q_1, \dots, Q_p$ are all defined. ∎

In Lemma 3.6, for each supervertex z there exists some $q(z)$ for which we can find a large set U_z that is super-$(m, q(z), q(z))$-linked to z. Here $q(z)$ may vary with z. With some more work, we can strengthen the result so that each supervertex z is linked to a large set by internally disjoint paths with prescribed constant lengths (that do not depend on z). We can then use this to show that there are many well-linked pairs of vertices. This is described in the next lemma.

LEMMA 3.8. *Let c, ϵ, D be positive reals, where $\epsilon \leq 0.5$ and $c \geq 22$. Let p, q_1, q_2, R be positive integers, where $\lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil + \lfloor \frac{2.5}{\epsilon} \rfloor \leq R \leq q_1 \leq q_2$ and q_1, q_2 have the same parity. Let n be a positive integer that is sufficiently large as a function of ϵ, p, q_1, q_2. Let G be an n-vertex bipartite graph satisfying* (a) $\delta(G) \geq cn^\epsilon$, (b) $\Delta(G) \leq Dn^\epsilon$, *and*

(c) *G contains no $(\frac{c}{2^7}, \epsilon)$-dense subgraph of radius at most R. Then there exist at least $n^{1.51}$ pairs of vertices that are (p, q_1, q_2)-linked.*

Proof. Let (X, Y) be a bipartition of G. Let $p^* = p(\frac{q_2 - q_1}{2} + 1)$. Let $m = p^*(\lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil)$, and let $k = 4p^* q_2$. For large enough n, by Lemma 3.6, there exist at least $n^{0.74}$ supervertices.

CLAIM 1. *For each supervertex z, there is a set S_z of at least $n^{0.8}$ vertices such that, for each $u \in S_z$, u and z are (p, q_1, q_2)-linked.*

Proof. Let z be any supervertex. By Lemma 3.6, for some positive integer $q(z) \le \lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil$ depending on z there is a set U_z of $n^{0.6}$ vertices that is super-$(m, q(z), q(z))$-linked to z. Let $r = q(z)$. Note that $m \ge p^* r$. For each $x \in U_z$ there exists a collection $\mathcal{P}_x$ of m internally disjoint x,z-paths of length r that are internally disjoint from U_z. Let F be the graph formed by taking the union of the paths in $\bigcup_{x \in U_z} \mathcal{P}_x$. Then F has radius at most $r \le \lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil \le R - \lfloor \frac{2.5}{\epsilon} \rfloor$, and $n(F) \le m(\lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil)|U_z|$. For large enough n, we can apply Lemma 3.4 with $S = U_z$, F, k, r defined as above, and $\gamma_\epsilon = m(\lfloor \frac{4.3}{\epsilon} \rfloor + \lceil \frac{1}{4\epsilon} \rceil)$, to obtain sets/levels $L_1, \dots, L_{R-r+1}$ outside $V(F) - U_z$, where $L_1 = U_z$. Note that the L_i's alternate between being contained in X and being contained in Y. To apply the last statement of Lemma 3.4 with $b = 0.6$, we need to check that $R - r \ge l(0.6, \epsilon)$. We have $l(0,6,\epsilon) = \lceil \frac{\ln(0.4/0.19)}{0.8\epsilon} \rceil + \lceil \frac{0.2}{0.6}(\frac{1}{\epsilon} + 1) \rceil \le \lceil \frac{0.94}{\epsilon} \rceil + \lceil \frac{1}{3\epsilon} \rceil + 1 \le \frac{2.5}{\epsilon}$, where the last inequality can be obtained by considering whether $\epsilon < \frac{1}{3}$ or $\frac{1}{3} \le \epsilon \le 0.5$. Since $R - r \ge \lfloor \frac{2.5}{\epsilon} \rfloor$ by our earlier discussion, we have $R - r \ge l(0.6, \epsilon)$. By the last statement of Lemma 3.4, for some positive integer $t \le l(0.6, \epsilon) + 1 \le \lfloor \frac{2.5}{\epsilon} \rfloor + 1$, L_t is a strong level and $|L_t|, |L_{t-1}| \ge n^{0.81}$. Note that $r + (t-1) \le R$.

Let B denote the subgraph of G induced by $L_{t-1} \cup L_t$. Then B is bipartite with (L_{t-1}, L_t) being a bipartition. Since L_t is a strong level, when we constructed it in the proof of Lemma 3.4 we applied Case 1. Thus as in Case 1, $e(B) \ge \frac{3}{8}|L_{t-1}|\frac{d}{4} \ge \frac{3c}{32} n^{a_{t-1}+\epsilon}$, where $d \ge cn^\epsilon$ is the minimum degree of G. Also, by condition (c) each vertex x in L_t has neighbors in at least k different sectors of L_{t-1}. In particular, $d_B(x) \ge k$. So, $e(B) \ge k|L_s|$. Starting with B, we iteratively remove vertices whose degree becomes less than $k/4$ until no such vertex exists. Let B' denote the remaining subgraph of B. Since fewer than $kn(B)/4 = k(|L_{t-1}| + |L_t|)/4 \le k|L_t|/2 \le e(B)/2$ edges are removed in the process, $e(B') \ge e(B)/2 \ge \frac{3c}{64} n^{a_{t-1}+\epsilon} \ge n^{a_{t-1}+\epsilon}$, since $c \ge 22$. Let $L'_{t-1} = L_{t-1} \cap V(B')$ and $L'_t = L_t \cap V(B')$. Then (L'_{t-1}, L'_t) is a bipartition of B'. We have $|L'_t| \ge e(B')/\Delta(G) \ge n^{a_{s-1}+\epsilon}/Dn^\epsilon = |L_{t-1}|/D$. Similarly, $|L'_{t-1}| \ge |L_{t-1}|/D$. By our definition of B', $\delta(B') \ge k/4 = p^* q_2$.

Let $q'_1 = q_1 - r - (t-1)$ and $q'_2 = q_2 - r - (t-1)$. Since $q_2 \ge q_1 \ge R \ge r + (t-1)$, $q'_1, q'_2 \ge 0$. Since q_1, q_2 have the same parity, so do q'_1 and q'_2. Let $T = T(p, q'_1, q'_2)$ denote the spider with $p(\frac{q'_2 - q'_1}{2} + 1) = p^*$ legs such that, for each j between q'_1 and q'_2 and having the same parity as them, p of these legs have length j. It is possible that $q'_1 = 0$, in which case p of the p^* legs of T have length 0. Clearly, $e(T) < p^* q_2$. Since $\delta(B') \ge p^* q_2$, by Lemma 2.5, for any vertex u in $V(B')$ there is a copy $T(u)$ of T in B', where u is the image of the root.

We now define our set S_z as follows: If q'_1 is odd, we let $S_z = L'_{t-1}$. If q'_1 is even, we let $S_z = L'_t$. In either case, $|S_z| \ge |L_{t-1}|/D \ge n^{0.8}$ for large n. For each $u \in S_z$, the corresponding $T(u)$ has all of its leaves lying in $L'_t \subseteq L_t$. Since L_t is strong, each vertex in L'_t has neighbors in at least k different sectors in L_{t-1}. Since $k \ge p^*$, we can lengthen each leg of $T(u)$ by one to get a tree $T'(u)$ so that the leaves of $T'(u)$ all lie in different sectors of L_{t-1}. By repeatedly applying condition 4 of Lemma 3.4, we can find disjoint paths of length $t-2$ through $L_1 \cup L_2 \cup \cdots \cup L_{t-1}$ linking these leaves to different vertices of $L_1 = U_z$. Every two of these paths share only u in common. Denote this new spider by $T''(u)$. In forming $T''(u)$, each leg of $T(u)$ is lengthened by $t-1$. So for, each j between $q_1 - r$ and $q_2 - r$ having the same parity as them, p of the legs of $T''(u)$ have length j. Let $W \subseteq U_z$ denote the set of the p^* leaves of $T''(u)$. Note that $V(T''(u)) \cap V(F) = W$. Since U_z is super-(m, r, r)-linked to z in F and $m \geq p^* r$, by Lemma 3.7, we can find p^* paths of length r in F linking W to z such that every two of these paths share only z. The union of these paths with $T''(u)$ now yields a (p, q_1, q_2)-path system in G between z and u. This holds for each $u \in S_z$, and the claim is proved. □

Since there are at least $n^{0.74}$ supervertices, by Claim 1, the number of (p, q_1, q_2)-linked pairs in G is at least $\frac{1}{2} n^{0.74} \cdot n^{0.8} > n^{1.51}$ for large n. Thus we have shown the lemma. □

Now, we use Lemma 3.8 to find desired subdivided graphs in dense graphs that don't contain dense subgraphs of small radius. To use Lemma 3.8, we need the graph G to be "almost regular." We will use Proposition 2.7, introduced in section 2, to "reduce" a host graph to an almost regular one.

THEOREM 3.9. *Let c, ϵ be positive reals, where $\epsilon \leq \frac{1}{2}$ and $c \geq 110$. Let F be a subdivision of another graph H. For every edge $xy \in E(H)$, let $l_{x,y}$ denote the stretch of xy in F. Suppose that $l_{x,y}$ is even and $l_{x,y} \geq 2\lfloor \frac{8}{\epsilon} \rfloor$ for all $xy \in E(H)$. There is a function $n_0(\epsilon, F)$ of ϵ and F such that for all integers $n \geq n_0(\epsilon, F)$ if G is an n-vertex (c, ϵ)-dense bipartite graph that contains no $(\frac{c}{5 \cdot 2^7}, \epsilon)$-dense subgraph of radius at most $\lfloor \frac{8}{\epsilon} \rfloor$, then G contains a copy of F.*

Proof. Let $R = \lfloor \frac{8}{\epsilon} \rfloor$. By our assumption, G contains no $(\frac{c}{5 \cdot 2^7}, \epsilon)$-dense subgraph of radius at most R. Let $l = \max\{l_{x,y} : xy \in E(H)\}$. Let $q = l - R$. For each $xy \in E(H)$, let $q_{x,y} = l_{x,y} - R$. Then $R \leq q_{x,y} \leq q$, and $q_{x,y}$ has the same parity as R and q. Let $t = n(H)$. Let $H' = H^{(2)}$ denote the graph obtained from H by subdividing each edge of H exactly once. Then $H^{(2)}$ has fewer than t^2 vertices, and by Corollary 2.2, $ex(n, H^{(2)}) < t^2 n^{3/2}$.

For each $xy \in E(H)$, we may split the strict x,y-path in F into a path of length R and a path of length $q_{x,y}$. By doing so, we may view F as a subdivision of $H' = H^{(2)}$. For each edge $uv \in E(H')$, let $l'_{u,v}$ denote the stretch of uv in F. Then by our discussion in the previous paragraph, $l'_{u,v}$ is between R and q and has the same parity as R and q.

By removing edges if needed, we may assume that $e(G) = cn^{1+\epsilon}$. By Proposition 2.7, G contains a subgraph G' on $m \geq n^{\frac{\epsilon}{2} \frac{1-\epsilon}{1+\epsilon}}$ vertices such that $e(G') \geq \frac{2c}{5} m^{1+\epsilon}$ and $\frac{\Delta(G')}{\delta(G')} \leq c_\epsilon$, where $c_\epsilon = 20 \cdot 2^{\frac{1}{\epsilon^2}+1}$. Iteratively remove vertices of degree less than $\frac{c}{5} m^\epsilon$ from G' until we get stuck; call the remaining subgraph G''. By our procedure, $e(G'') \geq \frac{c}{5} m^{1+\epsilon}$ and $\delta(G'') \geq \frac{c}{5} m^\epsilon$. Note that in forming G'' from G', the minimum degree does not decrease. This is because either no vertex is removed at all or, if some vertex is removed, then $\delta(G') \leq \frac{c}{5} m^\epsilon$ whereas $\delta(G'') \geq \frac{c}{5} m^\epsilon$. Obviously, $\Delta(G'') \leq \Delta(G')$. Hence, $\frac{\Delta(G'')}{\delta(G'')} \leq c_\epsilon$ still holds. Let $N = n(G'')$. Clearly as n tends to infinity, m tends to infinity and N tends to infinity. Write $\delta(G'') = \frac{c}{5} N^{\epsilon'} = c' N^{\epsilon'}$, where $c' = \frac{c}{5} \geq 22$ and $\epsilon' \geq \epsilon$. Then $\Delta(G'') \leq c' c_\epsilon N^{\epsilon'}$. If $\epsilon' > \frac{1}{2}$, then for large enough N we can apply Corollary 2.2 to get $F \subseteq G''$. So we may assume that $\epsilon' \leq \frac{1}{2}$.

Since G contains no $(\frac{c}{5 \cdot 2^7}, \epsilon)$-dense subgraph of radius at most R, G'' has no $(\frac{c'}{2^7}, \epsilon')$-dense subgraph of radius at most R. By considering whether $\epsilon' < 0.25$ or $0.25 \leq \epsilon' \leq 0.5$, one can show $\lfloor \frac{4.3}{\epsilon'} \rfloor + \lceil \frac{1}{4\epsilon'} \rceil + \lfloor \frac{2.5}{\epsilon'} \rfloor \leq \frac{6.8}{\epsilon'} + \frac{0.5}{\epsilon'} < \frac{8}{\epsilon'}$. So, $R = \lfloor \frac{8}{\epsilon} \rfloor \geq \lfloor \frac{8}{\epsilon'} \rfloor \geq \lfloor \frac{4.3}{\epsilon'} \rfloor + \lceil \frac{1}{4\epsilon'} \rceil + \lfloor \frac{2.5}{\epsilon'} \rfloor$. Recall that $t = n(H)$. Let $p = t^2 q$. By Lemma 3.8, with ϵ' in place of ϵ, c' in place of c, $c'c_\epsilon$ in place of D, and R, q in place of q_1, q_2, respectively, G'' contains at least $N^{1.51}$ many (p, R, q)-linked pairs.

Define a new graph L with $V(L) = V(G'')$ such that $uv \in E(L)$ if and only if u and v are (p, R, q)-linked in G''. Then, $e(L) \geq N^{1.51} \geq t^2 N^{3/2} \geq ex(N, H')$. Thus, L contains a copy M of H'. Note that M has at most $2\binom{t}{2} < t^2$ edges. By our definition of L, for every edge $uv \in E(M) \subseteq E(L)$, vertices u and v are (p, R, q)-linked in G''. In particular, there are p internally disjoint u, v-paths of length $l'_{u,v}$ in G''. We can build a copy of F in G'' by replacing each $uv \in E(M)$ with a u, v-path $Q_{u,v}$ of length $l'_{x,y}$ in G''. If we can ensure that these paths $Q_{u,v}$ are pairwise vertex disjoint outside $V(M)$, then their union forms a copy of F. We can accomplish this, since when any pair (u, v) is processed fewer than $t^2 \cdot q = p$ vertices have been used to embed earlier pairs, and we can choose a u, v-path of length $l'_{u,v}$ that is internally disjoint from all these vertices. ∎

4. Proof of Theorem 1.10. In this section, we prove Theorem 1.10. Before we proceed to the main proof, we need one more lemma.

LEMMA 4.1. *Let k, n, R be positive integers, where $k, R \geq 2$. Let ϵ, M be positive reals, where $\epsilon < 1$ and $M \geq (2k)^R$. Suppose G is an n-vertex bipartite graph with average degree at least M and radius at most R. There exist a positive integer d, where $2 \leq d \leq R$, and disjoint independent sets $X, Y \in V(G)$ such that the following hold:*

1. *The subgraph H of G induced by $X \cup Y$ has average degree at least $\frac{M}{(2k)^R}$.*
2. *For any two vertices $a, b \in Y$ and a set $S \subseteq X \cup Y$ with $|S| \leq k-2$ there exist a neighbor a' of a and a neighbor b' of b in $X - S$ together with an a', b'-path of length $2d-2$ that is internally disjoint from $V(H)$.*

Proof. Let d denote the smallest positive integer $j \leq R$ such that G has a subgraph that has average degree at least $\frac{M(2k)^j}{(2k)^R}$ and radius j. Since G has average degree at least M and radius at most R, d is well-defined. Also, $d \geq 2$, since any radius-one subgraph of G is a star and has average degree less than $2 \leq \frac{M(2k)}{(2k)^R}$. Let F be a subgraph of G that has average degree at least $\frac{M(2k)^d}{(2k)^R}$ and radius d. Let u be a vertex in the center of F. For each $i = 0, 1, \ldots, d$, let L_i denote the set of vertices at distance i from u in F. Let T be a tree in F rooted at u with $V(T) = L_0 \cup L_1 \cup \cdots \cup L_{d-1}$. Let $v_1, \ldots, v_r$ denote the children of u in T. For each $j = 1, \ldots, r$, let T_j denote the subtree of T rooted at v_j. For each $j = 1, \ldots, r$, let $A_j = V(T_j) \cap L_{d-1}$ and $B_j = N_F(A_j) \cap L_d$. Let

$$L_d^+ = \{x \in L_d : x \text{ lies in} \geq k \text{ different } B_j's\},$$
$$L_d^- = \{x \in L_d : x \text{ lies in} < k \text{ different } B_j's\}.$$

CLAIM 2. *At least half of the edges in F between L_{d-1} and L_d are incident to L_d^+.*

Proof. Otherwise suppose that fewer than half of these edges are incident to L_d^+. Then $e(F - L_d^+) > \frac{1}{2}e(F) \geq \frac{1}{2}\frac{1}{2}\frac{M(2k)^d}{(2k)^R}n(F) = \frac{1}{4}\frac{M(2k)^d}{(2k)^R}n(F)$, using that F has average degree at least $\frac{M(2k)^d}{(2k)^R}$. For each $j = 1, \ldots, r$, let $B_j^- = B_j \cap L_d^-$, and let D_j denote the subgraph of F induced by $V(T_j) \cup u \cup B_j^-$. Clearly each D_j has radius at most $d-1$. So, by our definition of d, we must have

$$e(D_j) \leq \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}n(D_j). \quad (2)$$

Summing (2) over all $j = 1, \dots, r$, we have

$$\sum_{j=1}^{r} e(D_j) \le \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}\sum_{j=1}^{r} n(D_j). \tag{3}$$

By our definition of B_j and D_j, each vertex of $F - L_d^+$ lies in at most $k-1$ different D_j's, except that u lies in all $r \le n(F)$ of the D_j's. Hence (3) yields

$$e(F - L_d^+) \le \sum_{j=1}^{r} e(D_j) \le \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}[(k-1)n(F) + n(F)] = \frac{1}{4}\frac{M(2k)^d}{(2k)^R}n(F),$$

contradicting our earlier claim that $e(F - L_d^+) > \frac{1}{4}\frac{M(2k)^d}{(2k)^R}n(F)$. □

Now, since $F - L_d$ has radius at most $d-1$, by our assumption,

$$e(F - L_d) \le \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}n(F - L_d) \le \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}n(F).$$

Hence

$$e_F(L_{d-1}, L_d) = e(F) - e(F - L_d) \ge \left[\frac{1}{2}\frac{M(2k)^d}{(2k)^R} - \frac{1}{2}\frac{M(2k)^{d-1}}{(2k)^R}\right]n(F) \ge \frac{M}{(2k)^R}n(F).$$

Now, let $X = L_{d-1}$ and $Y = L_d^+$, and let H denote the subgraph of F induced by $X \cup Y$. By Claim 2, $e(H) \ge \frac{1}{2}e_F(L_{d-1}, L_d) \ge \frac{1}{2}\frac{M}{(2k)^R}n(F) \ge \frac{1}{2}\frac{M}{(2k)^R}n(H)$. So, H has average degree at least $\frac{M}{(2k)^R}$. To prove the second statement, let $a, b \in Y = L_d^+$ and $S \subseteq X \cup Y$, where $|S| \le k-2$. Since $a \in L_d^+$ and $|S| \le k-2$, we can find a neighbor a' of a in $X - S$. Suppose $a' \in A_i$. Since $b \in L_d^+$, we can find a neighbor b' in $X - S$ that belongs to A_j for some $j \neq i$. Since T_i and T_j are rooted at two different children of u, there is an a', b'-path P of length $2d-2$ through u that lies inside $L_0 \cup \cdots \cup L_{d-2} \cup \{a', b'\}$. The path P is internally disjoint from H. □

Remark 4.2. A subtle but crucial point of Lemma 4.1 is that the same integer d works for all pairs $a, b \in Y$ simultaneously. This fact will play an important role in our final proof below.

Proof of Theorem 1.10. Recall that F is the desired subdivision of another graph H. Let $R = \lfloor \frac{8}{\epsilon} \rfloor$. Clearly, $R \ge 2$. For each edge $xy \in E(H)$, let $l_{x,y}$ denote the stretch of xy in F; by our assumption, $l_{x,y}$ is even and $l_{x,y} \ge 2R$. Suppose $\max\{l_{x,y} : xy \in E(H)\} = 2q$. Let $t = n(H)$ and $k = t(t-1)q + 2$. Let $c = c_F = 110n_0(\epsilon, F)qt^2(5 \cdot 2^7)^{t(t-1)}(2k)^{Rt(t-1)}$, where $n_0(\epsilon, F)$, as specified in Theorem 3.9, depends only on ϵ and F. Let G be an n-vertex graph with at least $cn^{1+\epsilon}$ edges. We show $F \subseteq G$. For convenience, we may assume that $H = K_t$; otherwise for each pair $x, y \in V(H)$, where $xy \notin E(H)$, we may set $l_{x,y} = 2R$. The resulting subdivision F' of K_t contains F and also satisfies the conditions of Theorem 1.10 (noting that the constant c_F depends only on ϵ and t). Hence we may as well assume that $H = K_t$ and $F = F'$.

Let G' be a spanning bipartite subgraph of G with $e(G') \ge \frac{1}{2}e(G)$. It is well known that G' exists. For each integer $i \ge 0$, let $c_i = \frac{c}{(5\cdot 2^7)^i(2k)^{iR}}$ and let $c_i' = c_i/(5 \cdot 2^7)$. Then $c_i'/(2k)^R = c_{i+1}$. Let $H_0 = G'$ and $n_0 = n(H_0)$. We iteratively define a sequence of subgraphs of G' as follows. Since G' is bipartite, all these graphs will be bipartite. If such a subgraph exists, let G_1 be a (c_0', ϵ)-dense subgraph of H_0 that has radius at most R. Then G_1 has average degree at least $2c'_0[n(G_1)]^\epsilon$. By Lemma 4.1, there exist a positive integer $d_1 \le R$ and disjoint independent sets X_1, Y_1 in $V(G_1)$ such that the subgraph H_1 of G_1 induced by $X_1 \cup Y_1$ has average degree at least $2c'_0[n(G_1)]^\epsilon/(2k)^R \ge 2c_1[n(H_1)]^\epsilon$. So, H_1 is (c_1, ϵ)-dense. Furthermore, for every pair of vertices u_1, v_1 in Y_1 and a set $S_1 \subseteq X_1$ with $|S_1| \le k-2$, there exist a neighbor u'_1 of u_1 and a neighbor v'_1 of v_1 together with a u'_1, v'_1-path in G_1 of length $2d_1 - 2$ that is internally disjoint from H_1.

In general, suppose we have defined $G_1, H_1, G_2, H_2, \ldots, G_i, H_i$. If such a subgraph exists, let G_{i+1} be a (c'_i, ϵ)-dense subgraph of H_i that has radius at most R. Then G_{i+1} has average degree at least $2c'_i[n(G_{i+1}]^\epsilon$. By Lemma 4.1, there exist a positive integer $d_{i+1} \le R$ and disjoint independent sets X_{i+1}, Y_{i+1} in $V(G_{i+1})$ such that the subgraph H_{i+1} of G_{i+1} induced by $X_{i+1} \cup Y_{i+1}$ has average degree at least $2c'_i[n(G_{i+1})]^\epsilon/(2k)^R \ge 2c_{i+1}[n(H_{i+1})]^\epsilon$. Thus, H_{i+1} is (c_{i+1}, ϵ)-dense. Furthermore, for every pair of vertices u_{i+1}, v_{i+1} in Y_{i+1} and a set $S_{i+1} \subseteq X_{i+1}$ with $|S_{i+1}| \le k-2$ there exist a neighbor u'_{i+1} of u_{i+1} and a neighbor v'_{i+1} of v_{i+1} together with a u'_{i+1}, v'_{i+1}-path in G_i of length $2d_{i+1} - 2$ that is internally disjoint from H_{i+1}.

Let $s = t(t-1)$ if we can carry on this process for $t(t-1)$ steps without getting stuck. Otherwise, let s be the largest index $i < t(t-1)$ such that G_i, H_i are defined but no subgraph G_{i+1} of H_i exists that fits the description. We consider two cases.

Case 1. $s = t(t-1)$. We describe how to find a copy of F in G as follows. Let $T_{q,t}$ denote the tree obtained from a path $P = a_1 b_1 a_2 b_2 \ldots a_t b_t$ on $2t$ vertices by attaching $t-1$ (labelled) paths of length q at each vertex of P. Then $T_{q,t}$ has $q(t-1)2t + 2t \le 2qt^2$ vertices. Since H_s is (c_s, ϵ)-dense, H_s has average degree has at least $2c_s[n(H_s)]^\epsilon$. So, H_s contains a subgraph with minimum degree at least $c_s[n(H_s)]^\epsilon \ge c_s \ge 2qt^2$, by our definition of c and the fact that $s = t(t-1)$. By Lemma 2.5, this subgraph contains a copy of $T = T_{q,t}$. Since $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_s$, T lies in all of $H_1, \ldots, H_s$. For each $i \in \{1, \ldots, s\}$, $(X_i \cap V(T), Y_i \cap V(T))$ is a bipartition of T. We must have either all of $a_1, \ldots, a_t$ contained in X_i or all of $b_1, \ldots, b_t$ contained in X_i. Without loss of generality, we may assume that for at least half of the indices i in $\{1, \ldots, s\}$ the former holds (otherwise we can rename the a_i's with the b_i's and the b_i's with the a_i's). By skipping indices if necessary, we may assume for each $i \in \{1, \ldots, \frac{t(t-1)}{2}\}$ that all of $a_1, \ldots, a_t$ are contained in X_i (and thus all of $b_1, \ldots, b_t$ are contained in Y_i).

We will now find in G a copy of F with $b_1, \ldots, b_t$ as the branch vertices. Let f be a bijection between $\binom{[t]}{2}$ and $\{1, \ldots, \frac{t(t-1)}{2}\}$. For all pairs $i, j \in \{1, \ldots, t\}$, where $i < j$, we do the following. Let $m = f(i,j)$. In T there are $t-1$ (labelled) paths of length q attached to b_i. Let $z_{i,j}$ denote the vertex on the jth path whose distance from b_i along the path is $l_{i,j} - 2d_m$. Let $P_{i,j}$ denote the portion of this path between b_i and $z_{i,j}$. Since $b_i \in Y_m$ and $l_{i,j} - 2d_m$ is even, $z_{i,j} \in Y_m$. Let $T^* = \bigcup_{1 \le i < j \le t(t-1)/2} P_{i,j}$. Note that $V(T^*) \subseteq V(T) \subseteq X_m \cup Y_m$ for each $m \in \{1, \ldots, \frac{t(t-1)}{2}\}$. Next, we complete T^* into a copy of F by adding disjoint paths $Q_m \subseteq G_m$ of length $2d_m$ connecting $z_{i,j}$ to b_j.

We define Q_m inductively in the decreasing order of $m = f(i,j)$ (where $i < j$) satisfying (a) $Q_m \subseteq G_m$, (b) Q_m joins $z_{i,j}$ to b_j and has length $2d_m$, and (c) Q_m intersects $V(T^*) \cup \bigcup_{r=m+1}^{t(t-1)/2} Q_r$ only in $z_{i,j}$ and b_j. For the basis step, let $m = \frac{t(t-1)}{2}$. Let i, j, where $i < j$, be the pair with $f(i,j) = m$. By our earlier discussion, $z_{i,j}, b_j \in Y_m$. Let $S = V(T^*)$. Then $|S| \le t(t-1)q = k-2$. By Lemma 4.1, we can find a neighbor $z'_{i,j}$ and a neighbor b'_j of b_j in $X_m - S$ together with a $z'_{i,j}, b'_j$-path Q'_m of length $2d_m - 2$ in G_m that is internally disjoint from H_m. Let $Q_m =$

$Q'_m \cup z_{i,j}z'_{i,j} \cup b_jb'_j$. Then Q_m joins $z_{i,j}$ and b_j and has length $2d_m$. Furthermore, since $S \subseteq V(H_m)$ and Q'_m is internally disjoint from H_m, Q_m intersects S only in $z_{i,j}$ and b_j. For the induction step, let $m < \frac{t(t-1)}{2}$ and suppose that Q_r has been defined for $r = m+1, \dots, \frac{t(t-1)}{2}$ that satisfies the three conditions. Let i, j, where $i < j$, be the pair with $f(i,j) = m$. By our earlier discussion, $z_{i,j}, b_j \in Y_m$. Let $S = V(T^*) \cup \bigcup_{r=m+1}^{t(t-1)/2} V(Q_r)$. Since $Q_r \subseteq G_r \subseteq H_m$, $S \subseteq V(H_m)$. Also, $|S| \le t(t-1)q = k-2$. By Lemma 4.1, we can find a neighbor $z'_{i,j}$ and a neighbor b'_j of b_j in $X_m - S$ together with a $z'_{i,j}, b'_j$-path Q'_m of length $2d_m - 2$ in G_m that is internally disjoint from H_m. Let $Q_m = Q'_m \cup z_{i,j}z'_{i,j} \cup b_jb'_j$. Then Q_m joins $z_{i,j}$ and b_j and has length $2d_m$. Furthermore, since $S \subseteq V(H_m)$ and Q'_m is internally disjoint from H_m, Q_m intersects S only in $z_{i,j}$ and b_j. This completes the construction of the Q_m's and Case 1.

Case 2. $s < t(t-1)$. In this case, by our assumption, H_s is (c_s, ϵ)-dense but has no (c'_s, ϵ)-dense subgraph of radius at most R (otherwise G_{i+1} would have been defined). In other words, H_s has no $(\frac{c_s}{5\cdot 2^7}, \epsilon)$-dense subgraph of radius at most R. Also, clearly $c_s \ge 110$ and $n(H_s) \ge c_s > n_0(\epsilon, t)$ by our definition of c and $s < t(t-1)$. By Theorem 3.9, $F \subseteq H_s$ and hence $F \subseteq G$. This completes Case 2 and the proof of the theorem. ∎

5. Further discussions. As mentioned in the introduction, our main goal in the paper is to establish an $O(n^{1+\frac{c}{p}})$ bound on $ex(n, K_t^{(p)})$. Currently we have $c = 16$. A more careful analysis can reduce c by quite a bit. However, to substantially reduce c, it seems like new ideas are needed. A possible new approach to explore is the one used by Fox and Sudakov [11].

Appendix A: Proof of Proposition 1.2.

PROPOSITION 1.2. *Let F be a bipartite graph with m vertices and e edges. Then $ex(n, F) = \Omega(n^{2-\frac{m}{e}+\frac{1}{e}})$. More generally, we have $ex(n, F) = \Omega(n^{2-\frac{1}{\gamma}})$, where $\gamma = \gamma(F)$ is the local-density of F.*

Proof. Since $ex(n, F) \ge ex(n, H)$ for any $H \subseteq F$, it suffices to prove the first statement. Let n be sufficiently large as a function of m and e. Let $p = \frac{1}{4}n^{-\frac{m-1}{e}}$. Consider the random graph $G = G(n, p)$ (that is, edges are included in G independently with probability p). Fixing any set S of $\frac{n}{2}$ vertices, the number of edges $e(G[S])$ of G induced by S has the binomial distribution $BIN(\frac{n}{2}, p)$. By Chernoff's inequality $Prob(e(G[S]) < \frac{n^2p}{16}) < 2e^{-\lambda n^2}$ for some constant λ depending on p (and thus on m and e). Since there are $\binom{n}{n/2} < 2^n$ many choices of S, the probability that, for some set S of $\frac{n}{2}$ vertices, $G[S]$ has fewer than $\frac{n^2p}{16}$ edges is less than $2^n \cdot 2e^{-\lambda n^2}$. In particular, for sufficiently large n, this probability is less than $\frac{1}{2}$.

Let X be the random variable that counts the number of copies of F in G. There are fewer than n^m potential labelled copies of F on $V(G)$, each being a subgraph of G with probability p^e. Hence $E(X) < n^m p^e < (1/4)n^m(n^{-\frac{m-1}{e}})^e = \frac{n}{4}$. By Markov's inequality, $Prob(X > \frac{n}{2}) < \frac{1}{2}$.

By our discussions above, with positive probability G satisfies that (a) for every set S of $\frac{n}{2}$ vertices, $e(G[S])$ has at least $\frac{n^2p}{16}$ edges, and (b) the number of labelled copies of F in G is at most $\frac{n}{2}$. Fix such an n-vertex graph G. If we delete one vertex from each copy of F in G, we are left with a subgraph G' of G on at least $\frac{n}{2}$ vertices that has no copy of F. By (a), G' has at least $\frac{n^2p}{16} = \Omega(n^{2-\frac{m-1}{e}})$ edges. By adding isolated vertices to G' if necessary, we see that $ex(n, F) = \Omega(n^{2-\frac{m-1}{e}})$. ∎

Appendix B: Proof of Proposition 2.7.

PROPOSITION 2.7. *Let ϵ, c be positive reals, where $\epsilon < 1$ and $c \geq 1$. Let n be a positive integer that is sufficiently large as a function of ϵ. Let G be an n-vertex graph with $e(G) \geq cn^{1+\epsilon}$. Then G contains a subgraph G' on $m \geq n^{\frac{\epsilon}{2}\frac{1-\epsilon}{1+\epsilon}}$ vertices such that $e(G') \geq \frac{2c}{5}m^{1+\epsilon}$ and $\frac{\Delta(G')}{\delta(G')} \leq c_\epsilon$, where $c_\epsilon = 20 \cdot 2^{\frac{1}{\epsilon^2}+1}$.*

Proof. For convenience, we will drop ceilings and floors whenever doing so does not affect the analysis in an essential way. We say that a graph H is *d-good* if $\frac{\Delta(H)}{\delta(H)} \leq d$. Let ϵ, c be positive reals, where $\epsilon < 1$ and $c \geq 1$. Let n be a positive integer sufficiently large as a function of ϵ. Let G be a graph on n vertices with $e(G) \geq cn^{1+\epsilon}$. Set $p = \lceil 2^{\frac{1}{\epsilon^2}+1} \rceil$. We partition $V(G)$ into $2p$ almost equal parts $B_1, \ldots, B_{2p}$, where B_1 consists of $\lceil \frac{n}{2p} \rceil$ vertices of the highest degrees in G.

Suppose first that at most $\frac{c}{2}n^{1+\epsilon}$ edges of G are incident to B_1. We say that G is of *type* 1. Let $H = G - B_1$. Then $e(H) \geq \frac{c}{2}n^{1+\epsilon}$. Successively remove vertices of degree less than $\frac{c}{10}n^\epsilon$ from H until we get stuck; denote the remaining subgraph by G'. Let $m = n(G')$. Since at most $\frac{c}{10}n^\epsilon \cdot n = \frac{c}{10}n^{1+\epsilon}$ edges are removed in the process, we have $e(G') \geq \frac{4c}{10}n^{1+\epsilon} \geq \frac{2c}{5}m^{1+\epsilon}$. Also, $\delta(G') \geq \frac{c}{10}n^\epsilon$ by the way we obtained G'. By our assumption of B_1, $d_G(x) \geq \Delta(G')$ for all $x \in B_1$. Also, $\sum_{x \in B_1} d_G(x) \leq cn^{1+\epsilon}$ since at most $\frac{c}{2}n^{1+\epsilon}$ edges of G are incident to B_1. We have $\Delta(G')(n/2p) \leq \sum_{x \in B_1} d_G(x) \leq cn^{1+\epsilon}$, from which we get $\Delta(G') \leq 2pcn^\epsilon$. Thus, $\Delta(G')/\delta(G') \leq 2pcn^\epsilon / \frac{c}{10}n^\epsilon = 20p$. So G' is $20(2^{\frac{1}{\epsilon^2}+1})$-good. Also, $m \geq 2e(G')/\Delta(G') \geq \frac{4c}{5}n^{1+\epsilon}/2pcn^\epsilon = \frac{2}{5p}n \geq n^{\frac{\epsilon}{2}\frac{1-\epsilon}{1+\epsilon}}$ for large n. So, the claim holds.

Suppose now that more than $\frac{c}{2}n^{1+\epsilon}$ edges of G are incident to B_1. We say that G is of *type* 2. By an averaging argument, for some $j \in \{2, \ldots, 2p\}$, the subgraph G_1 of G induced by $B_1 \cup B_j$ has more than $\frac{1}{2p}\frac{c}{2}n^{1+\epsilon} = \frac{c}{4p}n^{1+\epsilon}$ edges. Let $n_1 = n(G_1)$. Then $n_1 \approx \frac{n}{p}$. Note that $cn_1^{1+\epsilon} = c(\frac{n}{p})^{1+\epsilon} = \frac{c}{p}n^{1+\epsilon}\frac{1}{p^\epsilon} \leq \frac{c}{4p}n^{1+\epsilon}$, using that $p^\epsilon \geq 2^{(\frac{1}{\epsilon^2}+1)\cdot\epsilon} \geq 4$. So $e(G_1) \geq cn_1^{1+\epsilon}$.

We can now replace G with G_1 and repeat the analysis. If G_1 is of type 1, we terminate. If G_1 of type 2, we define G_2 from G_1 the way we defined G_1 from G. We continue like this as long as the new graph G_i is of type 2. We terminate when G_i is of type 1 for the first time. With $G_0 = G$, let k be the smallest i such that G_i is of type 1. Then $n(G_k) \approx \frac{n}{p^k}$ and $e(G_k) \geq \frac{c}{(4p)^k}n^{1+\epsilon}$. Since $e(G_k) \leq [n(G_k)]^2$, we have $\frac{c}{(4p)^k}n^{1+\epsilon} \leq \frac{n^2}{p^{2k}}$. From this, we get $(\frac{p}{4})^k \leq \frac{1}{c}n^{1-\epsilon} \leq n^{1-\epsilon}$ and hence $k \leq (1-\epsilon)\log n / \log \frac{p}{4}$. Since $n_k = n(G_k) \approx n/p^k$, $\log n_k \geq (1 - (1-\epsilon)\frac{\log p}{\log \frac{p}{4}})\log n$. Plugging in $p = 2^{\frac{1}{\epsilon^2}+1}$, we get $n_k \geq n^{\epsilon\frac{1-\epsilon}{1+\epsilon}}$. Since G_k is of type 1, by our earlier arguments, it contains a subgraph G' on m vertices, where $m \geq \frac{2}{5p}n_k \geq n^{\frac{\epsilon}{2}\frac{1-\epsilon}{1+\epsilon}}$ for large n. Furthermore, $e(G') \geq \frac{2c}{5}m^{1+\epsilon}$, and G' is $20(2^{\frac{1}{\epsilon^2}+1})$-good. This completes the proof. ∎

REFERENCES

[1] N. ALON, M. KRIVELEVICH, AND B. SUDAKOV, *Turán numbers of bipartite graphs and related Ramsey-type questions*, Combin. Probab. Comput., 12 (2003), pp. 477–494.

[2] B. BOLLOBÁS, *Extremal Graph Theory*, Academic Press, London, New York, 1978.

[3] A. BONDY AND M. SIMONOVITS, *Cycles of even length in graphs*, J. Combin. Theory Ser. B, 16 (1974), pp. 97–105.

[4] P. ERDŐS, *Some unsolved problems in graph theory and combinatorial analysis*, Combinatorial Mathematics and Its Applications (Proc. Conf. Oxford, 1969), Academic Press, London, 1971, pp. 97–109.

[5] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp, *On cycle-complete graph Ramsey numbers*, J. Graph Theory, 2 (1978), pp. 53–64.
[6] P. Erdős, *Graph theory and probability*, Canadian J. Math, 11 (1959), pp. 34–38.
[7] P. Erdős, *Extremal problems in graph theory*, in Theory of Graphs and Its Applications (Proc. Symp., Simolenice, 1963), New York, Academic Press, 1965, pp. 29–36.
[8] P. Erdős and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kut. Int. Közl., 5 (1960), pp. 17–61.
[9] P. Erdős and M. Simonovits, *Some extremal problems in graph theory*, in Combinatorial Theory and Its Applications 1 (Proc. Colloq. Balatonfüred, 1969), North–Holland, Amsterdam, 1970, pp. 377–390.
[10] R. Faudree and M. Simonovits, *On a class of degenerate extremal graph problems*, Combinatorica, 3 (1983), pp. 83–93.
[11] J. Fox and B. Sudakov, *Density theorems for bipartite graphs and related Ramsey-type results*, Combinatorica, 29 (2009), pp. 153–196.
[12] Z. Füredi, *On a Turán type problem of Erdős*, Combinatorica, 11 (1991), pp. 75–79.
[13] Z. Füredi, *Turán type problems*, in Surveys in Combinatorics, London Math. Soc. Lecture Note Ser. 166, Cambridge University Press, Cambridge, UK, 1991, pp. 253–300.
[14] T. Jiang, *Compact topological minors in graphs*, J. Graph Theory, 67 (2011), pp. 139–152.
[15] J. Kollár, L. Rónyai, and T. Szabó, *Norm graphs and bipartite Turán numbers*, Combinatorica, 16 (1996), pp. 399–406.
[16] T. Kövári, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math., 3 (1954), pp. 50–57.
[17] A. Kostochka and L. Pyber, *Small topological complete subgraphs of "dense" graphs*, Combinatorica, 8 (1988), pp. 83–86.
[18] F. Lazebnik, V. A. Ustimenko, and A. J. Woldar, *A new series of dense graphs of high girth*, Bull. Amer. Math. Soc., 32 (1995), pp. 73–79.
[19] W. Mader, *Homomorphieeigenschaften und mittlere Kantendichte von Graphen*, Math. Ann., 174 (1967), pp. 265–268.
[20] O. Pikhurko, *A note on the Turán function of even cycles*, Proc. Amer. Math. Soc, 140 (2012), pp. 3687–3992.
[21] B. Sudakov, *Recent developments in extremal combinatorics: Ramsey and Turán type problems*, Proc. Internat. Congress Mathemat. India, 4 (2010), pp. 2579–2606.
[22] A. Thomason, *The extremal function for complete minors*, J. Combin. Theory Ser. B, 81 (2001), pp. 318–338.
[23] J. Verstraëte, *On arithmetic progressions of cycle lengths in graphs*, Combin. Probab. Comput., 9 (2000), pp. 369–373.
[24] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 2001.