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EXACT SOLUTION OF THE HYPERGRAPH TURÁN PROBLEM FOR K-UNIFORM LINEAR PATHS

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A k-uniform linear path of length ℓ, denoted by $\mathbb{P}_\ell^{(k)}$, is a family of k-sets $\{F_1,\dots,F_\ell\}$ such that $|F_i\cap F_{i+1}|=1$ for each i and $F_i\cap F_j=\emptyset$ whenever $|i-j|>1$. Given a k-uniform hypergraph H and a positive integer n, the *k-uniform hypergraph Turán number* of H, denoted by $\mathbf{ex}_k(n,H)$, is the maximum number of edges in a k-uniform hypergraph $\mathcal{F}$ on n vertices that does not contain H as a subhypergraph. With an intensive use of the delta-system method, we determine $\mathbf{ex}_k(n,P_\ell^{(k)})$ exactly for all fixed $\ell\ge 1, k\ge 4$, and sufficiently large n. We show that

$$\mathbf{ex}_k\left(n,\mathbb{P}_{2t+1}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\dots+\binom{n-t}{k-1}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set of t vertices. We also show that

$$\mathbf{ex}\left(n,\mathbb{P}_{2t+2}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\dots+\binom{n-t}{k-1}+\binom{n-t-2}{k-2},$$

and describe the unique extremal family. Stability results on these bounds and some related results are also established.

1. Introduction

As usual, a hypergraph $\mathcal{F}=(V,E)$ consists of a set V of vertices and a set E of edges, where each edge is a subset of V. We call edges of $\mathcal{F}$ *members*

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of $\mathcal{F}$. If each member of $\mathcal{F}$ is a k-subset of V, we say that $\mathcal{F}$ is a *k-uniform hypergraph* or a *k-uniform set system*. If $|V|=n$, it is often convenient to just let $V=[n]=\{1,\dots,n\}$. For convenience, we write $\mathcal{F}\subseteq\binom{[n]}{k}$ to indicate that $\mathcal{F}$ is a k-uniform hypergraph on vertex set $[n]$. There is a long history in the study of extremal problems concerning hypergraphs. Early well-known results include the Erdős-Ko-Rado theorem that says that for all $n>2k$ the maximum size of a k-uniform family on n vertices in which every two members intersect is $\binom{n-1}{k-1}$, with equality achieved by taking all the subsets of $[n]$ containing a fixed element. Given a family $\mathcal{H}$ of hypergraphs, the *k-uniform hypergraph Turán number* of $\mathcal{H}$, denoted by $\mathbf{ex}_k(n,\mathcal{H})$, is the maximum number of edges in a k-uniform hypergraph $\mathcal{F}$ on n vertices that does not contain a member of $\mathcal{H}$ as a subhypergraph. An $\mathcal{H}$-free family $\mathcal{F}\subseteq\binom{[n]}{k}$ is called *extremal* if $|\mathcal{F}|=\mathbf{ex}_k(n,\mathcal{H})$. If $\mathcal{H}$ consists of a single hypergraph H, we write $\mathbf{ex}_k(n,H)$ for $\mathbf{ex}_k(n,\{H\})$. If we let $M_2^{(k)}$ denote the k-uniform hypergraph consisting of two disjoint k-sets, then the Erdős-Ko-Rado theorem says $\mathbf{ex}_k(n,M_2^{(k)})=\binom{n-1}{k-1}$ for all $n>2k$. More generally, Erdős showed

Theorem 1.1. (Erdős [6]) *Let k,t be positive integers. There exists a number $n(k,t)$ such that for all integers $n>n(k,t)$, if $\mathcal{F}\subseteq\binom{[n]}{k}$ contains no $t+1$ pairwise disjoint members then*

$$|\mathcal{F}|\le\binom{n}{k}-\binom{n-t}{k}.$$

Furthermore, the only extremal family $\mathcal{F}$ consists of all the k-sets of $[n]$ meeting some fixed set S of t elements of $[n]$.

Surveys on Turán problems of graphs and hypergraphs can be found in [13] and [20]. Hypergraph Turán problems are notoriously difficult. The asymptotics are determined for very few hypergraphs and exact results are particularly rare. Most exact results concern specific hypergraphs on a small number of vertices (and often for fixed small values of k). For example, the exact value of $\mathbf{ex}_k(n,H)$ is determined (for large n) for the Fano plane, expanded triangle, 4-books with 2 pages, 4-books with 3 pages, 4-books with 4 pages, some 3-graphs with independent neighborhoods, extended complete graphs, generalized fans, and a couple of others (see [20] for details and references). By comparison, our results in this paper establish the exact value for every hypergraph in an infinite family (and for all $k\ge4$). In this regard, the exact result on extended complete graphs [26] (refining [24]) is similar in nature. However, the hypergraphs H we consider are much more sparse and more "spread out". So, our result may be viewed the first of its kind.

2. The Hypergraph problem for paths and main results

In this paper, we focus on the hypergraph problem for paths. As explained at the end of the previous section, the "spread out" nature of a path distinguishes the problem from most of the hypergraph Turán problems that have been studied. For $k=2$, the problem was solved by Erdős and Gallai in the following classic theorem.

Theorem 2.1. (Erdős-Gallai [5]) *Let G be a graph on n vertices containing no path of length ℓ. Then $e(G) \le \frac{1}{2}(\ell-1)n$. Equality holds iff G is the disjoint union of complete graphs on ℓ vertices.*

For $k \ge 3$, the most general definition of a k-uniform path is that of a Berge path. A *Berge path* of length ℓ is a family of distinct sets $\{F_1,\dots,F_\ell\}$ and $\ell+1$ distinct vertices $v_1,\dots,v_{\ell+1}$ such that for each $1 \le i \le \ell$, F_i contains v_i and v_{i+1}. Let $\mathcal{B}_\ell^{(k)}$ denote the family of k-uniform Berge paths of length ℓ. Győri et al. determined $\mathbf{ex}_k(n,\mathcal{B}_\ell^{(k)})$ exactly for infinitely many n.

Theorem 2.2. (Győri et al. [16]) *If $\ell > k \ge 2$ then $\mathbf{ex}_k(n,\mathcal{B}_\ell^{(k)}) \le \frac{n}{\ell}\binom{\ell}{k}$. Furthermore, equality is attained if ℓ divides n. If $3 \le \ell \le k$, then $\mathbf{ex}_k(n,\mathcal{B}_\ell^{(k)}) = \frac{n(\ell-1)}{k+1}$. Furthermore, here equality is attained if $k+1$ divides n.*

For the $\ell > k$ case, equality is attained by partitioning the n vertices into sets of size ℓ and taking a complete k-uniform hypergraph on each of the ℓ-set. For the $3 \le \ell \le k$ case, equality is attained by partitioning the n vertices into sets of size $k+1$ and taking exactly $\ell-1$ of the k-sets in each of these $(k+1)$-sets. The case $\ell=2$, $\mathbf{ex}_k(n,\mathcal{B}_2^{(k)}) = \lfloor n/k \rfloor$, is obvious.

A notion that is more restrictive than a Berge path is that of a loose path. A *loose path* of length ℓ is a family of sets $\{F_1,\dots,F_\ell\}$ such that $F_i \cap F_j \neq \emptyset$ iff $|i-j|=1$. Let $\mathcal{P}_\ell^{(k)}$ denote the family of k-uniform loose paths of length ℓ.

Theorem 2.3. (Mubayi-Verstraëte [25]) *Let $k,\ell \ge 3$, $t=\lfloor(\ell-1)/2\rfloor$ and $n \ge (\ell+1)k/2$. Then $\mathbf{ex}_k(n,\mathcal{P}_3^{(k)}) = \binom{n-1}{k-1}$. For $\ell,k>3$, we have*

$$t\binom{n-1}{k-1} + O\left(n^{k-2}\right) \le \mathbf{ex}_k\left(n,\mathcal{P}_\ell^{(k)}\right) \le 2t\binom{n-1}{k-1} + O\left(n^{k-2}\right).$$

An even more restrictive notion than that of a loose path is the notion of a linear path. A *linear path* of length ℓ is a family of sets $\{F_1,\dots,F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ for each i and $F_i \cap F_j = \emptyset$ whenever $|i-j|>1$. Let $\mathbb{P}_\ell^{(k)}$ denote the k-uniform linear path of length ℓ. It is unique up to isomorphisms. The determination of $\mathbf{ex}_k(n,\mathbb{P}_\ell^{(k)})$ is nontrivial even for $\ell=2$. This was solved by Frankl [9] (see [22] for more on the $k=4$ case). The case $\ell<k$ was asymptotically determined in [10]. As the main result of this paper, we determine $\mathbf{ex}_k(n,\mathbb{P}_\ell^{(k)})$ exactly, for **all** fixed k,ℓ, where $k\geq 4$, and sufficiently large n.

Theorem 2.4. **(Main result)** *Let k,t be positive integers, $k\geq 4$. For sufficiently large n, we have*

$$\mathbf{ex}_k\left(n,\mathbb{P}_{2t+1}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{n-t}{k-1}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements. Also,

$$\mathbf{ex}\left(n,\mathbb{P}_{2t+2}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{n-t}{k-1}+\binom{n-t-2}{k-2}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements plus all the k-sets in $[n]\setminus S$ that contain some two fixed elements.

Our method does not quite work for the $k=3$ case. We conjecture that a similar result holds for $k=3$. Using essentially the same method (for $k\geq 4$) and a slight modification of the method (for $k=3$), one can also determine the Turán numbers of loose paths for all fixed $k\geq 3$ and large n.

Theorem 2.5. *Let k,t be positive integers, where $k\geq 3$. For sufficiently large n, we have*

$$\mathbf{ex}_k\left(n,\mathcal{P}_{2t+1}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{n-t}{k-1}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t vertices. Also,

$$\mathbf{ex}\left(n,\mathcal{P}_{2t+2}^{(k)}\right)=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{n-t}{k-1}+1.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t vertices plus one additional k-set that is disjoint from S.

Since Theorem 2.5 is not our main result and for $k \geq 4$ the proof is essentially the same as that of Theorem 2.4, we will not formally prove Theorem 2.5. We will instead just briefly comment on how to prove Theorem 2.5 at the end of Section 5. For details, see [18].

It is easy to see that the constructions described in the above two theorems are indeed $\mathbb{P}_\ell^{(k)}$ and $\mathcal{P}_\ell^{(k)}$-free, respectively. We will show that for large enough n they are the unique extremal constructions for the respective Turán numbers.

We organize our paper as follows. In Section 3, we introduce our main tool: the delta-system method and develop some useful facts. In Section 4, we establish asymptotically tight bounds. In Section 5, we prove the exact bounds, characterize the extremal families and establish stability results. In Section 6, we prove a related result. In Section 7 we collect a few problems and remarks.

3. The delta-system method and homogeneous families

The *delta-system method*, started by Deza, Erdős and Frankl [3] and others, is a powerful tool for solving set system problems. The method is summarized in a structural lemma obtained by the first author [12] (see Lemma 3.1 below). It has been used successfully to obtain a series of sharp results on set systems, most notable in [10], and more recently in [15].

We now introduce a few definitions. A family of sets $F_1, \dots, F_s$ are said to form an *s-star* or *Δ-system* of size s with *kernel* A if $F_i \cap F_j = A$ for all $1 \leq i < j \leq s$. Sets $F_1, \dots, F_s$ are called the *petals* (or *members*) of the Δ-system. Given a family $\mathcal{F}$ of sets and a member F of $\mathcal{F}$, we define the *intersection structure* of F relative to $\mathcal{F}$ to be

$$\mathcal{I}(F, \mathcal{F}) = \{F \cap F' \colon F' \in \mathcal{F}, F' \neq F\}.$$

In other words, $\mathcal{I}(F, \mathcal{F})$ consists of all the intersections of F with other members of $\mathcal{F}$. As in many k-uniform hypergraph problems, it is often convenient to assume the family $\mathcal{F}$ to be k-partite. A k-uniform family $\mathcal{F} \subseteq \binom{[n]}{k}$ is *k-partite* if there exists a partition of the vertex set $[n]$ into k sets $X_1, \dots, X_k$, called *parts*, such that $\forall F \in \mathcal{F}$ and $\forall i \in [k]$ we have $|F \cap X_i| = 1$. So, each member of $\mathcal{F}$ consists of one vertex from each part. We will call $(X_1, \dots, X_k)$ a (vertex) *k-partition* of $\mathcal{F}$. Recall that an old result of Erdős and Kleitman [8] showed that every k-uniform family $\mathcal{H}$ contains a k-partite subfamily $\mathcal{H}' \subseteq \mathcal{H}$ of size at least $(k!/k^k)|\mathcal{H}|$.

Let $\mathcal{F}\subseteq\binom{[n]}{k}$ be a k-partite family with a k-partition $(X_1,\dots,X_k)$. Given any subset $S\subseteq[n]$, its *pattern*, denoted by $\Pi(S)$, is defined as

$$\Pi(S)=\{i\colon S\cap X_i\neq\emptyset\}\subseteq[k].$$

In other words, the pattern of S records which parts in the given k-partition that S meets. If $\mathcal{L}$ is a collection of subsets of $[n]$, then we define

$$\Pi(\mathcal{L})=\{\Pi(S)\colon S\in\mathcal{L}\}\subseteq 2^{[k]}.$$

We will call $\Pi(\mathcal{I}(F,\mathcal{F}))$ the *intersection pattern* of F relative to $\mathcal{F}$.

Lemma 3.1. (The intersection semilattice lemma [12]) *For any positive integers s and k, there exists a positive constant $c(k,s)$ such that* **every** *family $\mathcal{F}\subseteq\binom{[n]}{k}$ contains a subfamily $\mathcal{F}^*\subseteq\mathcal{F}$ satisfying*

1. *$|\mathcal{F}^*|\geq c(k,s)|\mathcal{F}|$.*
2. *$\mathcal{F}^*$ is k-partite, together with a k-partition $(X_1,\dots,X_k)$.*
3. *There exists a family $\mathcal{J}$ of proper subsets of $[k]$ such that $\Pi(\mathcal{I}(F,\mathcal{F}^*))=\mathcal{J}$ holds for all $F\in\mathcal{F}^*$.*
4. *$\mathcal{J}$ is closed under intersection, i.e., for all $A,B\in\mathcal{J}$ we have $A\cap B\in\mathcal{J}$ as well.*
5. *Fixing any $F\in\mathcal{F}^*$, for each $A\in\mathcal{I}(F,\mathcal{F}^*)$ there exists an s-star in $\mathcal{F}^*$ containing F with kernel A.*

Note that for $s\geq k$, item 4 follows from items 3 and 5. For $s<k$, observe that if all items hold for $s=k$, then they certainly also hold for all $s<k$.

Definition 3.2. We call family $\mathcal{F}^*$ that satisfies items (2)-(5) of Lemma 3.1 a (k,s)-*homogeneous family* with intersection pattern $\mathcal{J}$. When the context is clear we will drop the (k,s)-prefix.

A useful notion in the delta-system method is the notion of a rank of family. Given a family $\mathcal{L}$ of subsets of $[k]$, we define the *rank* of $\mathcal{L}$, denoted by $r(\mathcal{L})$ as

$$r(\mathcal{L})=\min\{|D|\colon D\subseteq[k],\nexists B\in\mathcal{L},D\subseteq B\}.$$

So, $r(\mathcal{L})$ is the cardinality of a smallest set D that "obstructs" $\mathcal{L}$ in the sense that no member of $\mathcal{L}$ contains it. We will apply the rank notion to the intersection pattern $\mathcal{J}\subsetneq 2^{[k]}$. If $\mathcal{F}$ is a k-partite family with a k-partition $(X_1,\dots,X_k)$, $F\in\mathcal{F}$ and $D\subseteq[k]$, we will let $F[D]=F\cap(\bigcup_{i\in D}X_i)$. That is, $F[D]$ is the projection of F onto the parts whose indices are in D. Given a family $\mathcal{F}\subseteq\binom{[n]}{k}$ and a subset $W\subseteq[n]$, we define the *degree* of W in $\mathcal{F}$ as

$$\deg_{\mathcal{F}}(W)=|\{F\colon F\in\mathcal{F},W\subseteq F\}|.$$

Lemma 3.3. *Let k,s be positive integers. Let $\mathcal{F}^*$ be a (k,s)-homogeneous family with intersection pattern $\mathcal{J}$. Let $D \subseteq [k]$. Suppose no member of $\mathcal{J}$ contains D. Then* $\deg_{\mathcal{F}^*}(F[D]) = 1$.

Proof. Suppose that F' is another member of $\mathcal{F}^*$ besides F that contains $F[D]$. Then $F \cap F' \supseteq F[D]$. Let $B = \Pi(F \cap F')$. Then $B \supseteq D$. Since $\mathcal{F}^*$ is homogeneous with intersection pattern $\mathcal{J}$, $B \in \mathcal{J}$. This contradicts our assumption that no member of $\mathcal{J}$ contains D. ∎

Lemma 3.3 immediately implies

Proposition 3.4. (The rank bound) *Let k,s be positive integers. Let $\mathcal{F}^*$ be a (k,s)-homogeneous family on n vertices with intersection pattern $\mathcal{J}$. If $r(\mathcal{J}) = p$, then $|\mathcal{F}^*| \leq \binom{n}{p}$.*

Proof. By definition, $\exists D \subseteq [k]$ with $|D| = p$ such that no member of $\mathcal{J}$ contains D. By Lemma 3.3, for each $F \in \mathcal{F}^*$, $F[D]$ is a p-subset of F that is not contained in any other member of $\mathcal{F}^*$. Suppose $F_1, \ldots, F_m$ are all the members of $\mathcal{F}^*$. Then $F_1[D], F_2[D], \ldots, F_m[D]$ are all distinct p-sets, and clearly there can be at most $\binom{n}{p}$ of them. So, $|\mathcal{F}^*| = m \leq \binom{n}{p}$. ∎

In the spirit of Proposition 3.4, we will focus on homogeneous families whose intersection patterns $\mathcal{J}$ have rank k–1 or k. Among rank k–1 patterns, we consider two types.

Definition 3.5. Let $\mathcal{L}$ be a family of proper subsets of $[k]$ that has rank $k-1$. We say that $\mathcal{L}$ is of type 1 if there exists an element $x \in [k]$ such that $[k] \setminus \{x\} \notin \mathcal{L}$ but $\forall y \in [k], y \neq x, [k] \setminus \{y\} \in \mathcal{L}$. If $\mathcal{L}$ has rank $k-1$, but is not of type 1, then we say that it is of type 2.

We now prove some quick facts.

Lemma 3.6. *Let $k \geq 3$ be a positive integer. Let $\mathcal{L}$ be a family of proper subsets of $[k]$ that is closed under intersection.*

1. *If $\mathcal{L}$ has rank k, then it consists of all the proper subsets of $[k]$.*
2. *If $\mathcal{L}$ has rank $k-1$ and is of type 1, then for some $i \in [k]$, $\mathcal{L}$ contains all the proper subsets of $[k]$ that contain i. We will call i the central element.*
3. *For $k \geq 4$, if $\mathcal{L}$ has rank $k-1$ and is of type 2 then $\mathcal{L}$ contains at least two singletons.*

Proof. First, assume that $\mathcal{L}$ has rank k. By the definition of rank, every $(k-1)$-subset of $[k]$ belongs to $\mathcal{L}$. Since $\mathcal{L}$ is closed under intersection, every proper subset of $[k]$ is in $\mathcal{L}$.

Next, suppose that $\mathcal{L}$ has rank $k-1$ and is of type 1. By definition, there exists $i\in[k]$ such that $[k]\setminus\{i\}\notin\mathcal{L}$ but $\forall j\in[k], j\neq i, [k]\setminus\{j\}\in\mathcal{L}$. Since $\mathcal{L}$ is closed under intersection it contains all the proper subsets of $[k]$ that contain i.

Finally, assume that $\mathcal{L}$ has rank $k-1$ and is of type 2. By definition, there are some $t\geq 2$ different $(k-1)$-subsets of $[k]$ that obstruct $\mathcal{L}$. Without loss of generality, we may assume that $\forall i=1,\dots,t,[k]\setminus\{i\}\notin\mathcal{L}$ and $\forall i=t+1,\dots,k,[k]\setminus\{i\}\in\mathcal{L}$.

Claim. $\forall i,j\leq t, i\neq j$, we have $[k]\setminus\{i,j\}\in\mathcal{L}$.

Proof of Claim. Otherwise suppose for some $i,j\leq t, i\neq j$, $D=[k]\setminus\{i,j\}\notin\mathcal{L}$. Since $r(\mathcal{L})=k-1>k-2$, there must be some member of $\mathcal{L}$ that contains D. However, the only possible members of $\mathcal{L}$ that could contain D are $[k]\setminus\{i\}$ and $[k]\setminus\{j\}$, neither of which is in $\mathcal{L}$, a contradiction.

By our discussions above, we know $\forall i=t+1,\dots,k,[k]\setminus\{i\}\in\mathcal{L}$ and $\forall i,j\in[t],[k]\setminus\{i,j\}\in\mathcal{L}$ and $\mathcal{L}$ is closed under intersection. If $t\geq 3$, then $\{i\}\in\mathcal{L}$ for each $i\in[t]$. If $t=2$, then $\{i\}\in\mathcal{L}$ for each $i\in\{3,\dots,k\}$. So, in particular, if $k\geq 4$ then $\mathcal{L}$ contains at least two singletons. ∎

Lemma 3.6 immediately yields

Corollary 3.7. *Let k,s be positive integers, where $s\geq k\geq 4$. Let $\mathcal{F}^*$ be a (k,s)-homogeneous family with intersection pattern $\mathcal{J}$. Suppose $\mathcal{J}$ has rank k or has rank $k-1$ and is of type 2. Let $F\in\mathcal{F}^*$. Then there exist at least two distinct vertices $u,v\in F$ such that $\{u\}$ is the kernel of some s-star in $\mathcal{F}^*$ and $\{v\}$ is the kernel of some s-star in $\mathcal{F}^*$.*

Proof. By Lemma 3.6(3), there exist $i,j\in[k]$ such that $\{i\}\in\mathcal{J}$ and $\{j\}\in\mathcal{J}$. Let $u=F[\{i\}]$ and $v=F[\{j\}]$. Since $\mathcal{F}^*$ is homogeneous, $\{u\}=F[\{i\}]\in\mathcal{I}(F,\mathcal{F}^*)$ and $\{v\}=F[\{j\}]\in\mathcal{I}(F,\mathcal{F}^*)$. By Lemma 3.1(5), each of $\{u\}$ and $\{v\}$ is the kernel of some s-star in $\mathcal{F}^*$. ∎

A hypergraph (set system) $\mathcal{H}$ is *linear* if every two members of $\mathcal{H}$ intersect in at most one vertex. Given a graph H, the *k-blowup*, denoted by $[H]^{(k)}$ (or $H^{(k)}$ for short), is the k-uniform hypergraph obtained from H by replacing each edge xy in H with a k-set E_{xy} that consists of x,y and $k-2$ new vertices such that for distinct edges $xy,x'y'$, $(E_{xy}-\{x,y\})\cap(E_{x'y'}-\{x',y'\})=\emptyset$. If H has p vertices and q edges, then $H^{(k)}$ has $p+q(k-2)$ vertices and q hyperedges. The resulting $H^{(k)}$ is a k-uniform linear hypergraph whose vertex set contains the vertex set of H. We call H the *skeleton* of $H^{(k)}$.

We adopt the convention that P_ℓ denotes a path with ℓ edges (and $\ell+1$ vertices). Then $[P_\ell]^{(k)}$ is a k-uniform linear path of length ℓ. Throughout the paper, we denote this hypergraph by $\mathbb{P}_\ell^{(k)}$.

Theorem 3.8. *Let k,s,q be positive integers where $k \geq 4$ and $s \geq kq$. Let T be a q-edge tree. Let $\mathcal{F}^*$ be a (k,s)-homogeneous family with intersection pattern $\mathcal{J}$. If $\mathcal{J}$ has rank k or has rank $k-1$ and is of type 2, then $T^{(k)} \subseteq \mathcal{F}^*$.*

Proof. For convenience, if $\{x\}$ is the kernel of an s-star in $\mathcal{F}^*$ we call x a *kernel vertex* in $\mathcal{F}^*$. We use induction on q to find a copy of $T^{(k)}$ in $\mathcal{F}^*$ in which each vertex of $V(T)$ is mapped to a kernel vertex in $\mathcal{F}^*$. For the basis step, let $q=1$. So T consists of a single edge xy. We take any member $F \in \mathcal{F}^*$. By Corollary 3.7, there exist $u,v \in F$ that are kernel vertices in $\mathcal{F}^*$. Now, F is a copy of $T^{(k)}$. Furthermore, by mapping x to u and y to v, we fulfill the additional requirement that each vertex in $V(T)$ is mapped to a kernel vertex in $\mathcal{F}^*$. For the induction step, let $q \geq 2$. Let v be a leaf of T and u its unique neighbor in T. Let $T_1 = T - v$. By induction hypothesis, $\mathcal{F}^*$ contains a copy L of $[T_1]^{(k)}$ in which each vertex of $V(T_1)$ is mapped to a kernel vertex in $\mathcal{F}^*$. Suppose u is mapped to u'. Then $\{u'\}$ is the kernel of an s-star S in $\mathcal{F}^*$. Suppose $F_1, \ldots, F_s$ are the petals of S. Since $F_1 \setminus \{u'\}, \ldots, F_s \setminus \{u'\}$ are pairwise disjoint and $s \geq kq > |L|$, for some j, $F_j \setminus \{u'\}$ is disjoint from $L \setminus \{u'\}$. Now $L \cup F_j$ forms a copy of $T^{(k)}$. Furthermore, by Corollary 3.7, F_j contains some v' other than u' that is a kernel vertex in $\mathcal{F}^*$. By mapping v to v', we maintain the condition that each vertex of $V(T)$ is mapped to a kernel vertex in $\mathcal{F}^*$. This completes the proof. ∎

Theorem 3.9. *Let k,l,s be positive integers with $k \geq 4$ and $s \geq kl$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Suppose $\mathbb{P}_\ell^{(k)} \not\subseteq \mathcal{F}$. Then $\mathcal{F}$ can be partitioned into subfamilies $\mathcal{G}_1, \ldots, \mathcal{G}_m, \mathcal{F}_0$ such that $\forall i \in [m]$, $\mathcal{G}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ which has rank $k-1$ and type 1, and $|\mathcal{F}_0| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$.*

Proof. First we apply Lemma 3.1 to $\mathcal{F}$ to get a (k,s)-homogeneous subfamily $\mathcal{G}_1$ with intersection pattern $\mathcal{J}_1$ such that $|\mathcal{G}_1| \geq c(k,s)|\mathcal{F}|$. Then we apply Lemma 3.1 again to $\mathcal{F} - \mathcal{G}_1$ to get a homogeneous subfamily $\mathcal{G}_2$ with intersection pattern $\mathcal{J}_2$ such that $|\mathcal{G}_2| \geq c(k,s)(|\mathcal{F}| - |\mathcal{G}_1)$. We continue like this. Let m be the smallest nonnegative integer such that $\mathcal{J}_{m+1}$ has rank $k-2$ or less. Let $\mathcal{F}_0 = \mathcal{F} - (\bigcup_{i=1}^m \mathcal{G}_i)$. By our procedure, $|\mathcal{G}_{m+1}| \geq c(k,s)|\mathcal{F}_0|$. Since $\mathcal{J}_{m+1}$ has rank at most $k-2$, by Lemma 3.4, $|\mathcal{G}_{m+1}| \leq \binom{n}{k-2}$ and hence $|\mathcal{F}_0| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$.

By our assumption, $\mathcal{J}_1, \ldots, \mathcal{J}_m$ all have rank at least $k-1$. If for some i, either $\mathcal{J}$ has rank k or has rank $k-1$ and is of type 2, then by Theorem 3.8, $\mathbb{P}_\ell^{(k)} \subseteq \mathcal{G}_i \subseteq \mathcal{F}$, contradicting our assumption that $\mathbb{P}_\ell^{(k)} \not\subseteq \mathcal{F}$. So for each $i \in [m]$, $\mathcal{J}_i$ is of type 1. ∎

For the remaining sections, we will refer to the partition given in Theorem 3.9 as a *canonical partition* of $\mathcal{F}$.

4. Kernel graphs and asymptotic bounds

In this section, we introduce some auxiliary graphs associated with the given family $\mathcal{F} \subseteq \binom{[n]}{k}$. Using these we can quickly establish asymptotically tight bounds on $\mathbf{ex}_k(n, \mathbb{P}^{(k)}_{2t+1})$ and $\mathbf{ex}_k(n, \mathbb{P}^{(k)}_{2t+2})$. Some of the definitions and lemmas in this section may be of independent interests. Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$ and a subset $W \subseteq [n]$, we define the *kernel degree* of W, denoted by $\deg^*_{\mathcal{F}}(W)$, as

$$\deg^*_{\mathcal{F}}(W) = \max\{s\colon \exists \text{ an } s\text{-star with kernel } W \text{ in } \mathcal{F}\}.$$

Note that the *kernel degree* of W is a stronger notion than the degree of W.

Definition 4.1. Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$, the *kernel-graph* with *threshold* s is the graph L on $[n]$ such that $\forall x, y \in [n]$, $xy \in E(L)$ iff $\deg^*_{\mathcal{F}}(\{x, y\}) \geq s$.

Lemma 4.2. *Let H be a graph with q edges. Let $s = kq$. Let $\mathcal{F} \subseteq \binom{n]}{k}$. Let L be the kernel graph of $\mathcal{F}$ with threshold s. If $H \subseteq L$, then $\mathcal{F}$ contains a copy of $H^{(k)}$ whose skeleton is H.*

Proof. Let $e_1, \dots, e_q$ be the edges of H. For each i, suppose the two endpoints of e_i are x_i and y_i. We will replace each e_i with a member E_i of $\mathcal{F}$ that contain x_i, y_i such that $E_1 \setminus \{x_1, y_1\}, \dots, E_q \setminus \{x_q, y_q\}$ are pairwise disjoint. Since $x_1 y_1 \in E(L)$, $\deg^*_{\mathcal{F}}(\{x_1, y_1\}) \geq s$. Let E_1 be any member of $\mathcal{F}$ that contains x_1 and y_1 and avoids all $x_2, y_2, \dots, x_q, y_q$. In general, suppose we have found $E_1, E_2, \dots, E_{i-1}$. Since $x_i y_i \in E(L)$, $\deg^*_{\mathcal{F}}(\{x_i, y_i\}) \geq s$ there exists an s-star S in $\mathcal{F}$ with kernel $\{x_i, y_i\}$. Let $F_1, \dots, F_s$ denote the petals of S. Since $F_1 \setminus \{x_i, y_i\}, \dots, F_s \setminus \{x_i, y_i\}$ are pairwise disjoint and $|(\bigcup_{j=1}^{i-1} E_j) \setminus \{x_i, y_i\}| < kq = s$, there exists an $h \in [s]$ such that $F_h \setminus \{x_i, y_i\}$ is disjoint from $(\bigcup_{j=1}^{i-1} E_j) \setminus \{x_i, y_i\}$. We can let $E_i = F_h$. We can continue till we find $E_1, \dots, E_q$ in $\mathcal{F}$ that meet the requirements. The system $\{E_1, \dots, E_q\}$ forms a copy of $H^{(k)}$ whose skeleton is H. ∎

Suppose $\mathcal{F}$ can be decomposed into $\mathcal{F}_1, \dots, \mathcal{F}_m$, where for each $i \in [m]$, $\mathcal{F}_i$ is homogeneous with intersection pattern $\mathcal{J}_i$, where $\mathcal{J}_i$ has rank $k-1$ and is of type 1. We define the (k,s)*-homogeneous kernel graph* of $\mathcal{F}$ as follows. Fix any $F \in \mathcal{F}$. Suppose $F \in \mathcal{F}_p$. By Lemma 3.6, $\mathcal{J}_p$ has a central element i such that all proper subsets of $[k]$ containing i are members of $\mathcal{J}_p$. We fix such an i for $\mathcal{J}_p$. We have $\{i\} \in \mathcal{J}_p$ and $\{i, i'\} \in \mathcal{J}_p$ for each $i' \in [k] \setminus \{i\}$. So $F[\{i\}] \in \mathcal{I}(F, \mathcal{F}_p)$ and $F[\{i, i'\}] \in \mathcal{I}(F, \mathcal{F}_p)$ for each $i' \in [k] \setminus \{i\}$. We denote $F[\{i\}]$ by $c(F)$ and call it the *central element* of F. Thus, we have $c(F) \in \mathcal{I}(F, \mathcal{F}_p)$ and $\{c(F), y\} \in \mathcal{I}(F, \mathcal{F}_p)$ for each $y \in F \setminus \{c(F)\}$. Since $\mathcal{F}_p$ is

(k,s)-homogeneous, we have $\deg^*_{\mathcal{F}_p}(\{c(F)\}) \geq s$ and $\deg^*_{\mathcal{F}_p}(\{c(F),y\}) \geq s$ for each $y \in F \setminus \{c(F)\}$. In particular, this implies that

$$\deg^*_{\mathcal{F}}(\{c(F)\}) \geq s \text{ and } \deg^*_{\mathcal{F}}(\{c(F),y\}) \geq s \text{ for } \forall y \in F \setminus \{c(F)\}.$$

We define the (k,s)-*homogeneous kernel* digraph H of $\mathcal{F}$ to be a directed graph on $[n]$ whose edges consist of all the ordered pairs $(c(F),y)$ over all $F \in \mathcal{F}$ and $y \in F \setminus c(F)$. Furthermore, we mark $c(F)$ for each $F \in \mathcal{F}$. Let H' denote the underlying simple undirected graph of H. Note that H' is a subgraph of the kernel graph of $\mathcal{F}$ with threshold s. Also, note that at least one of the two endpoints of each edge of H' is marked. In this section, we will only make use of H' instead of H.

Lemma 4.3. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Suppose that $\mathcal{F}$ can be partitioned into subfamilies $\mathcal{F}_1,\dots,\mathcal{F}_m$ such that for each $i=1,\dots,m$, $\mathcal{F}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ that has rank $k-1$ and is of type 1. Let H be the (k,s)-homogeneous kernel digraph of $\mathcal{F}$. Let H' be the underlying undirected simple graph of H. We have*

$$|\mathcal{F}| \leq \frac{e(H')}{k-1}\binom{n-2}{k-2}.$$

Proof. Consider the number q of pairs $(\{x,y\},F)$ where $F \in \mathcal{F}, x,y \in F$ and $\{x,y\} \in E(H')$. Each $F \in \mathcal{F}$ contributes exactly $k-1$ to q. On the other hand, for each unordered pair x,y trivially there are at most $\binom{n-2}{k-2}$ members of $\mathcal{F}$ that contain x,y. So, each $xy \in E(H')$ contributes at most $\binom{n-2}{k-2}$ to q. So, we have $(k-1)|\mathcal{F}| = q \leq e(H')\binom{n-2}{k-2}$. ∎

Now we are ready to establish asymptotic tight bounds on $\mathbf{ex}_k(n,\mathbb{P}^{(k)}_{2t+1})$ and $\mathbf{ex}_k(n,\mathbb{P}^{(k)}_{2t+2})$. We need the following classical result concerning the circumference of a graph.

Lemma 4.4. (Erdős and Gallai [5]) *If G is an n-vertex graph that contains no cycle of length at least c, where $c \geq 3$, then $e(G) \leq \frac{1}{2}(c-1)(n-1)$.*

Theorem 4.5. *Let k,t be positive integers, where $k \geq 4$. We have*

$$\mathbf{ex}_k\left(n,\mathbb{P}^{(k)}_{2t+1}\right) \leq \mathbf{ex}_k\left(n,\mathbb{P}^{(k)}_{2t+2}\right) \leq t\binom{n-1}{k-1} + O\left(n^{k-2}\right).$$

Proof. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that contains no copy of $\mathbb{P}^{(k)}_{2t+2}$. Let $s = k(2t+2)$. By Theorem 3.9, there exists a partition of $\mathcal{F}$ into $\mathcal{G}_1,\dots,\mathcal{G}_m,\mathcal{F}_0$, where $|\mathcal{F}_0| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$ and for each $i \in [m]$ $\mathcal{G}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ that has rank $k-1$ and is of type 1. Let $\mathcal{F}' = \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_m$. Let H be the (k,s)-kernel digraph of $\mathcal{F}'$ and H' the underlying undirected simple graph of H.

Claim 1. H' has circumference at most $2t$.

Proof of Claim 1. For contradiction suppose H' contains a cycle C of length at least $2t+1$. Recall that in each edge of H', at least one endpoint is marked. If C has length at least $2t+2$, then we can find a path of length $2t+1$ on C with one of the endpoints being marked. If C has length $2t+1$ (which is odd) then we can find two consecutive vertices on C that are marked, in which case we can find a path of length $2t$ both of whose endpoints are marked.

In the former case, suppose $x_1x_2\ldots x_{2t+2}$ is a path of length $2t+1$ on C where x_1 is marked. By Lemma 4.2, $\mathcal{F}$ contains a copy $\mathcal{P}$ of $\mathbb{P}^{(k)}_{2t+1}$ whose skeleton is $x_1x_2\ldots x_{2t+2}$. Since x_1 is marked, $\deg^*_{\mathcal{F}'}(\{x_1\}) \ge s = k(2t+2)$. Let $F_1,\ldots,F_s$ be the petals of an s-star in $\mathcal{F}$ with kernel $\{x_1\}$. Since $\mathcal{P}$ has fewer than $k(2t+1)$ vertices and $F_1\setminus\{x_1\},\ldots,F_s\setminus\{x_1\}$ are pairwise disjoint, for some $h\in[s]$, $F_h\setminus\{x_1\}$ is disjoint from $\mathcal{P}$. We can add F_h to $\mathcal{P}$ to form a copy of $\mathbb{P}^{(k)}_{2t+2}$, contradicting the assumption that $\mathcal{F}$ contains no $\mathbb{P}^{(k)}_{2t+2}$.

In the latter case, suppose $x_1x_2\ldots x_{2t+1}$ is a path of length $2t$ on C where both x_1 and x_{2t+1} are marked. By Lemma 4.2, $\mathcal{F}$ contains a copy $\mathcal{P}$ of $\mathbb{P}^{(k)}_{2t}$ whose skeleton is $x_1x_2\ldots x_{2t+1}$. Using that $\deg^*_{\mathcal{F}'}(x_1) \ge s$ and $\deg^*_{\mathcal{F}'}(x_{2t+1}) \ge s$, we can extend $\mathcal{P}$ into a copy of $\mathbb{P}^{(k)}_{2t+2}$, a contradiction. ∎

Since H' has circumference at most $2t$, by Lemma 4.4 (with $c=2t+1$), we have $e(H') \le t(n-1)$. By Lemma 4.2, $|\mathcal{F}'| \le \frac{t(n-1)}{k-1}\binom{n-2}{k-2} = t\binom{n-1}{k-1}$. Therefore, $|\mathcal{F}| \le t\binom{n-1}{k-1} + \frac{1}{c(k,s)}\binom{n}{k-2}$. ∎

Note that if we were to just prove $\mathbf{ex}_k(n,\mathbb{P}^{(k)}_{2t+1}) \le t\binom{n-1}{k-1} + O(n^{k-2})$, it would have sufficed to just use the Erdős-Gallai theorem on $\mathbf{ex}(n,P_{2t+1})$ to get $e(H') \le tn$, from which the bound follows.

To close this section, we observe that following the arguments in [16], by iteratively removing vertices of degree at most $(k-1)(\ell-1)\binom{n-2}{k-2}$ one can prove by induction that the following bound holds for every n:

$$\mathbf{ex}(n,\mathbb{P}^{(k)}_{\ell}) \le (k-1)(\ell-1)\binom{n-1}{k-1}. \tag{1}$$

Even though this is a weaker bound than Theorem 4.5 for large n, it holds for every n. We will use this bound in certain estimates in the next section.

5. Proof of Theorem 2.4 and the stability of the bounds

In this section, we determine the exact value of $\mathbf{ex}_k(n,\mathbb{P}_\ell^{(k)})$ for large n. For convenience, we let $f(n,k,t) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1}$ and $g(n,k,t) = \binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{n-t}{k-1}+\binom{n-t-2}{k-2}$. Here, $f(n,k,t)$ is the number of k-sets in $[n]$ that meet a fixed set S of t elements of $[n]$ and $g(n,k,t)$ is $f(n,k,t)$ plus the number of k-sets in $[n]\setminus S$ that contain some fixed set of two elements. We wish to show that for fixed k,t, where $k\geq 4$, $\mathbf{ex}_k(n,\mathbb{P}_{2t+1}^{(k)}) = f(n,k,t)$ and $\mathbf{ex}_k(n,\mathbb{P}_{2t+2}^{(k)}) = g(n,k,t)$. As pointed out in the introduction, the lower bounds are easy to observe. Note that $f(n,k,t) \geq t\binom{n-1}{k-1} - c_1 n^{k-2}$ and $g(n,k,t) \geq t\binom{n-1}{k-1} - c_2 n^{k-2}$ for some constants c_1,c_2 depending on k,t. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that contains no copy of $\mathbb{P}_{2t+2}^{(k)}$. We may assume that there exists a constant c_3, depending on k and t, such that $|\mathcal{F}| \geq t\binom{n-1}{k-1} - c_3 n^{k-2}$, since otherwise $|\mathcal{F}| \leq f(n,k,t)$ and $|\mathcal{F}| \leq g(n,k,t)$ already hold. As a key step, we first show that $\mathcal{F}$ must already have a structure very similar to the extremal construction.

Let $s=k(2t+2)$. Let $\mathcal{G}_1,\ldots,\mathcal{G}_m,\mathcal{F}_0$ be a canonical partition of $\mathcal{F}$, where $|\mathcal{F}_0| \leq \frac{1}{c(k,s)}\binom{n}{k-2}$ and for each $i\in[m]$, $\mathcal{G}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ that has rank $k-1$ and is of type 1. Let $\mathcal{F}'=\bigcup_{i=1}^m \mathcal{G}_i$. Let H be the (k,s)-kernel digraph of $\mathcal{F}'$. Let H' denote the underlying undirected simple graph of H. Then $e(H)\leq 2e(H')$ (since each $\{u,v\}$ in $E(H')$ corresponds to at most two directed edges between u and v, namely (u,v) and/or (v,u)). For each $x\in V(H)$, let $d^+(x)$ denote the out-degree of x in H. By Claim 1 of Theorem 4.5, H' has circumference at most $2t$ and so $e(H')\leq t(n-1)<tn$. Thus $\sum_{x\in V(H)} d^+(x) = e(H) \leq 2e(H') < 2tn$. Let $D=n^{1-\frac{3/2}{k-1}}$. Define

$$A = \{x\in V(H)\colon d^+(x)\leq D\}, \quad B = \{x\in V(H)\colon d^+(x) > D\}.$$

Let $\mathcal{F}_A$ denote the set of members F of $\mathcal{F}'$ whose central element $c(F)$ lies in A. By our definition of H, we have

$$|\mathcal{F}_A| \leq |A|\cdot\binom{D}{k-1} < n\cdot D^{k-1} = n^{k-\frac{3}{2}}.$$

Since $\sum_{x\in V(H)} d^+(x) < 2tn$, we have $|B| < 2tn/D = 2tn^{\frac{3/2}{k-1}}$. The subgraph of H' induced by B, denoted by $H'[B]$, also has circumference at most $2t$ and thus $e(H'[B])<t|B|<2t^2 n^{\frac{3/2}{k-1}}$. Let $\mathcal{F}_B$ denote the set of members of $\mathcal{F}'$

that contain edges of $H'[B]$. We have

$$|\mathcal{F}_B| \le e(H'[B])\binom{n-2}{k-2} < 2t^2 n^{k-2+\frac{3/2}{k-1}} \le 2t^2 n^{k-\frac{3}{2}}.$$

Let $\widetilde{\mathcal{F}} = \mathcal{F}' \setminus (\mathcal{F}_A \cup \mathcal{F}_B) = \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_A \cup \mathcal{F}_B)$. By our discussions above,

$$|\widetilde{\mathcal{F}}| \ge t\binom{n-1}{k-1} - O\left(n^{k-\frac{3}{2}}\right).$$

By our definition of $\mathcal{F}_A$ and $\mathcal{F}_B$, we have

$$\widetilde{\mathcal{F}} = \{F \in \mathcal{F}' \colon c(F) \in B, |F \cap B| = 1\}.$$

Let $\widetilde{H}$ denote the subgraph of H consisting of all edges going from B to A. Suppose $B = \{x_1, \dots, x_p\}$. For each $i \in [p]$, let $d_i = d^+_{\widetilde{H}}(x_i)$. Based on the definition of $\widetilde{\mathcal{F}}$ we have

$$|\widetilde{\mathcal{F}}| \le \sum_{i=1}^{p} \binom{d_i}{k-1}.$$

Thus,

$$\sum_{i=1}^{p} \binom{d_i}{k-1} \ge t\binom{n}{k-1} - O\left(n^{k-\frac{3}{2}}\right). \tag{2}$$

Since $d_i < n$ we get $p \ge t$. On the other hand, we have

$$\sum_{i=1}^{p} d_i = e(\widetilde{H}) \le e(H') < tn. \tag{3}$$

Without loss of generality, we may assume that $d_1 \ge d_2 \ge \dots \ge d_p$. Let $1 \le m \le p$. For each $j > m$, since $d_j \le d_m$, we have $\binom{d_j}{k-1} \le \frac{d_j}{d_m}\binom{d_m}{k-1}$ and hence $\sum_{i=1}^{p}\binom{d_i}{k-1} \le \sum_{i=1}^{m}\binom{d_i}{k-1} + \frac{\sum_{i=m+1}^{p} d_i}{d_m}\binom{d_m}{k-1}$.

Claim 2. We have $d_1, \dots, d_t \ge n - O(n^{\frac{1}{2}})$.

Proof of Claim 2. Let $y = d_t$. By (3) and the observation above with $m = t-1$, we have

$$\begin{aligned} \sum_{i=1}^{p} \binom{d_i}{k-1} &\le \sum_{i=1}^{t-1} \binom{d_i}{k-1} + \frac{\sum_{i=t}^{p} d_i}{y}\binom{y}{k-1} \\ &\le \sum_{i=1}^{t-1} \binom{d_i}{k-1} + \frac{tn - \sum_{i=1}^{t-1} d_i}{y}\binom{y}{k-1}. \end{aligned} \tag{4}$$

Recall that for $i=1,\dots,t-1, y\le d_i\le n$. It is easy to see that the right-hand side of (4) is at most $(t-1)\binom{n}{k-1}+\frac{tn-(t-1)n}{y}\binom{y}{k-1}$. (Indeed, if $d_i<n$ for some $i\le t-1$, then by increasing d_i by 1 the net change would be $\binom{d_i}{k-2}-\frac{1}{y}\binom{y}{k-1}$ which is nonnegative.) This observation and (2) yield

$$t\binom{n}{k-1}-O\left(n^{k-\frac{3}{2}}\right)\le\sum_{i=1}^{p}\binom{d_i}{k-1}$$
$$\le(t-1)\binom{n}{k-1}+\frac{tn-(t-1)n}{y}\binom{y}{k-1}.$$

Rearranging, we get $\binom{y}{k-1}\frac{n}{y}\ge\binom{n}{k-1}-O(n^{k-\frac{3}{2}})$. A standard calculation yields $y\ge n-O(n^{\frac{1}{2}})$. ∎

Let us consider again the kernel graph L with threshold $s=(2t+2)k$, which is a 2-graph whose edges are pairs $\{u,v\}$ with kernel degree at least s in $\mathcal{F}$. By Claim 2, each of $x_1,\dots,x_t$ has out-degree at least $n-O(n^{\frac{1}{2}})$ in $\widetilde{H}$. So each of $x_1,\dots,x_t$ has degree at least $n-O(n^{\frac{1}{2}})$ in $H'\subseteq L$. So, in L the number of vertices that are adjacent to all of $x_1,\dots,x_t$ is at least $n-O(tn^{\frac{1}{2}})$, which is $n-O(n^{\frac{1}{2}})$. Let $S=\{x_1,\dots,x_t\}$ and let $W\subseteq[n]-S$ be the maximum set of vertices that are adjacent to all of S in L. By our discussion above, we have

Claim 3. $|W|\ge n-O(n^{\frac{1}{2}})$. ∎

Let

$$\mathcal{F}_S=\{F\in\binom{[n]}{k}\colon F\cap S\neq\emptyset\}.$$

We have $|\mathcal{F}_S|=f(n,k,t)$. Let $Z=V\setminus(S\cup W)$ and $n_1=|Z|$. Define

$$\mathcal{F}_1=\{F\in\mathcal{F}\colon F\subseteq Z\},\quad \mathcal{D}=\{F\in\binom{[n]}{k}\colon |F\cap S|=|F\cap Z|=1, F\notin\mathcal{F}\}.$$

By Claim 3 we have $n_1=O(n^{1/2})$. By (1) we have

$$|\mathcal{F}_1|\le k(2t+2)\binom{n_1-1}{k-1}. \tag{5}$$

Let $z\in Z$. By the definitions of W and L there exists an $x\in S$ such that $xz\notin E(L)$ and thus $\deg^*_{\mathcal{F}}(x,z)<s$. This means that the $(k-2)$-uniform family

$$\{F\setminus\{x,z\}\colon\{x,z\}\subseteq F\in\mathcal{F},\ |F\cap W|=k-2\}$$

contain no s pairwise disjoint members, so its size is at most $s\binom{|W|}{k-3}$ by Theorem 1.1. Hence, $\deg_{\mathcal{D}}(x,z) \ge \binom{|W|}{k-2} - s\binom{|W|}{k-3}$ and

$$(6) \qquad |\mathcal{D}| \ge |Z| \times \left(\binom{|W|}{k-2} - s\binom{|W|}{k-3} \right) \ge \Omega\left(n_1 \cdot n^{k-2}\right).$$

We are ready to complete the proof of the odd case.

Claim 4. If $\mathcal{F}$ contains no copy of $\mathbb{P}^{(k)}_{2t+1}$, then $|\mathcal{F}| \le f(n,k,t)$. Furthermore, equality holds only if $\mathcal{F}$ consists of all the k-sets in $[n]$ that meet S.

Proof of Claim 4. First, we show that every member of $\mathcal{F}$ that is disjoint from S is contained in Z. Suppose otherwise. Then there is a member F of $\mathcal{F}$ that is disjoint from S and intersects W. Let y_1 be any element in $F \cap W$. Since L has all the edges from S to W and $|W|$ is large, one can find a path $Q = y_1x_1y_2x_2\dots y_tx_ty_{t+1}$ of length $2t$ in L, where $y_1,\dots,y_t \in W$, such that $Q \cap F = \{y_1\}$. Using the fact that for each adjacent pair u,v on Q, $\deg^*_{\mathcal{F}}(\{u,v\}) \ge s = k(2t+2)$, we can extend $F \cup Q$ into a copy of $\mathbb{P}^{(k)}_{2t+1}$, a contradiction.

By the definitions of $\mathcal{F}_S, \mathcal{F}_1, \mathcal{D}$ and our discussion above, we have $\mathcal{F} \subseteq (\mathcal{F}_S \backslash \mathcal{D}) \cup \mathcal{F}_1$. By Equations (5) and (6) and the fact that $n_1 = O(n^{\frac{1}{2}})$ we have

$$(7) \qquad f(n,k,t) - |\mathcal{F}| = |\mathcal{F}_S| - |\mathcal{F}| \ge |\mathcal{D}| - |\mathcal{F}_1| \ge \Omega(n_1 \cdot n^{k-2}).$$

In particular, we have $|\mathcal{F}| \le f(n,k,t)$. Furthermore, equality holds only if $|Z| = n_1 = 0$ and $\mathcal{F} = \mathcal{F}_S$. ∎

Now, we prove the even case.

Claim 5. If $\mathcal{F}$ contains no copy of $\mathbb{P}^{(k)}_{2t+2}$ then $|\mathcal{F}| \le g(n,k,t)$. Furthermore, equality holds only if $\mathcal{F}$ consists of all the k-sets in $[n]$ that meet S plus all the k-sets in $[n] \setminus S$ that contain two fixed elements.

Proof of Claim 5. In addition to sets $\mathcal{F}_S, \mathcal{F}_1$, and $\mathcal{D}$, we define

$$\mathcal{F}_2 = \{F \in \mathcal{F} \colon F \cap S = \emptyset, |F \cap W| \ge 2\},$$
$$\mathcal{F}_3 = \{F \in \mathcal{F} \colon F \cap S = \emptyset, |F \cap W| = 1\}.$$

Obviously

$$(8) \qquad \mathcal{F} \subseteq (\mathcal{F}_S \setminus \mathcal{D}) \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.$$

Next, we obtain upper bounds on $|\mathcal{F}_2|$ and $|\mathcal{F}_3|$.

An r-intersecting family is a family of sets in which every two members intersect in at least r elements. Erdős-Ko-Rado showed [7] that for fixed k

and large n the unique largest r-intersecting family in $[n]$ is given by the family of all k-sets containing a fixed set of r elements (for $n > n(k,r)$).

We claim that $\mathcal{F}_2$ is a 2-intersecting family in $[n] \setminus S$. Otherwise, we can find two members E_1 and E_2 of $\mathcal{F}_2$ such that either $|E_1 \cap E_2| = 0$ or $|E_1 \cap E_2| = 1$. In the former case, we can find a path Q of length $2t$ in L using edges between S and W that meet E_1 and E_2 each at a single element. We can then extend $Q \cup E_1 \cup E_2$ into a copy of $\mathbb{P}_{2t+2}^{(k)}$, a contradiction. In the latter case, suppose $E_1 \cap E_2 = \{y\}$. Let w be an element in $(E_2 \cap W) \setminus \{y\}$. The element w exists, since $|E_2 \cap W| \geq 2$. We can find a path Q of length $2t$ in L between S and W that meets $E_1 \cup E_2$ only in w. Then we can extend $E_1 \cup E_2 \cup Q$ into a copy of $\mathbb{P}_{2t+2}^{(k)}$, again a contradiction.

We have shown that $\mathcal{F}_2$ is a 2-intersecting family in $[n] \setminus S$. By the Erdős-Ko-Rado theorem, (for $n > n_{k,t}$) we have

$$|\mathcal{F}_2| \leq \binom{n-t-2}{k-2}. \tag{9}$$

Furthermore, equality in Equation (9) holds only if $\mathcal{F}_2$ consists of all k-sets in $[n] \setminus S$ that meet two fixed elements $u, v \in [n] \setminus S$.

Now, consider $\mathcal{F}_3$. Let $\widehat{\mathcal{F}}_3 = \{F \setminus W : F \in \mathcal{F}_3\}$. Then $\widehat{\mathcal{F}}_3$ is a collection of $(k-1)$-sets in Z. For a member $C \in \widehat{\mathcal{F}}_3$, define the *multiplicity* of C to be the number of different $w \in W$ such that $C \cup w \in \mathcal{F}_3$. Let $\widehat{\mathcal{F}}_3'$ denote the set of members of $\widehat{\mathcal{F}}_3$ that have multiplicity 1 and $\widehat{\mathcal{F}}_3''$ the set of members of $\widehat{\mathcal{F}}_3$ that have multiplicity at least 2. Trivially, $|\widehat{\mathcal{F}}_3| \leq \binom{n_1}{k-1}$. We claim that $\widehat{\mathcal{F}}_3''$ must form an intersecting family. Otherwise suppose C_1, C_2 are two disjoint members of $\widehat{\mathcal{F}}_3''$. Since C_1, C_2 each has multiplicity at least 2, we can find $w_1, w_2 \in W, w_1 \neq w_2$, such that $E_1 = C_1 \cup w_1 \in \mathcal{F}_3$ and $E_2 = C_2 \cup w_2 \in \mathcal{F}_3$. Now, we can find a path Q of length $2t$ in L between w_1 and w_2 using edges of L between S and W such that Q intersects E_1 only in w_1 and E_2 only in w_2. Then we can extend $Q \cup E_1 \cup E_2$ into a copy of $\mathbb{P}_{2t+2}^{(k)}$ in $\mathcal{F}$, a contradiction. Since $\widehat{\mathcal{F}}_3''$ is an intersecting family in Z, by the Erdős-Ko-Rado theorem,

$$|\widehat{\mathcal{F}}_3''| \leq \min\left\{\binom{n_1-1}{k-2}, \binom{n_1}{k-1}\right\} \leq \binom{n_1}{k-2}.$$

Therefore

$$|\mathcal{F}_3| \leq |\widehat{\mathcal{F}}_3'| + |W||\widehat{\mathcal{F}}_3''| \leq \binom{n_1}{k-1} + n\binom{n_1}{k-2} \leq O\left(n \cdot n_1^{k-2}\right). \tag{10}$$

By (8), (5), (6), (9), (10), and the fact that $n_1 = O(n^{\frac{1}{2}})$, we have

$$\begin{aligned} |\mathcal{F}| &\le |\mathcal{F}_S| - |\mathcal{D}| + |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \\ &\le f(n,k,t) + \binom{n-t-2}{k-2} - \Omega\left(n_1 \cdot n^{k-2}\right) + O\left(n \cdot n_1^{k-2}\right) \\ &= g(n,k,t) - \Omega\left(n_1 \cdot n^{k-2}\right). \end{aligned} \tag{11}$$

In particular, we have

$$|\mathcal{F}| \le g(n,k,t).$$

Furthermore, equality holds only if $\mathcal{F}$ consists of all the members of $\mathcal{F}_S$ plus all the k-sets in $[n]$ that are disjoint from S and contain some two fixed elements u, v. ∎

With Claim 4 and Claim 5, we have completed the proof of Theorem 2.4. In addition, Equations (7) and (11) imply the following stability result on our bounds.

Theorem 5.1. *Let k,t be positive integers, where $k \ge 4$. Let ε be a small positive real. There exists a positive real δ and an integer n_0 such that for all integers $n \ge n_0$ if $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no copy of $\mathbb{P}_{2t+2}^{(k)}$ or $\mathbb{P}_{2t+1}^{(k)}$ and $|\mathcal{F}| \ge (t-\delta)\binom{n}{k-1}$ then there exists a set S of t elements in $[n]$ such that all except at most $\varepsilon\binom{n}{k-1}$ of the members of $\mathcal{F}$ intersect S.* ∎

To close the section, we briefly remark on how Theorem 2.5 is proved. For $k \ge 4$ and ℓ odd, Theorem 2.5 is implied by Theorem 2.4. For $k \ge 4$ and ℓ even, the proof is essentially the same except that we replace (9) with a simpler claim: there is at most one set that is disjoint from S and contained in W and otherwise $|\mathcal{F}_2| = 1 + O(n_1^{k-1})$ (since now we are just forbidding a loose path, instead of a linear path). For the $k = 3$ case, the approach is slightly different. We refer interested readers to [18].

6. Long linear paths vs. blow-ups of complete bipartite graphs

In this section, we describe a related result. First we prove a lemma. In our application of the lemma, we will choose m, n so that $m = o(n)$.

Lemma 6.1. *Let b, ℓ, q, t be positive integers, where $b \ge \binom{\ell}{t+1} \cdot q + \ell$. Let G be a bipartite graph with a bipartition (X, Y) where $|X| = m$, $|Y| = n$. Suppose $e(G) \ge bm + tn$. Then G contains either a copy of P_ℓ or a copy of $K_{t+1,q}$.*

Proof. We iteratively remove any vertex in X whose degree becomes less than b and any vertex in Y whose degree becomes at most t. We continue the process until no more vertex (from either X or Y) can be removed. Clearly fewer than $bm+tn$ edges are removed in the process. So the remaining subgraph G' is non-empty. By design, each vertex on G' in X has degree at least b and each vertex of G' in Y has degree at least $t+1$. Let Q be a longest path in G'. If Q has length at least ℓ then G contains P_ℓ and we are done. So we may assume that Q has length at most $\ell-1$.

Let v be an endpoint of Q and u its unique neighbor on Q. Since Q cannot be extended, we have $d_{G'}(v)\leq \ell-1$, which implies that $v\in Y$ and hence, $u\in X$. Since $d_{G'}(u)\geq b\geq \binom{\ell}{t+1}\cdot q+\ell$, u has at least $\binom{\ell}{t+1}\cdot q$ neighbors in G' that lie outside Q. None of them has a neighbor outside Q or else we get a path longer than Q, a contradiction. But each of them has at least $t+1$ neighbors in G' (all of which must lie on Q). By the pigeonhole principle, some q of them are adjacent to the same set of $t+1$ vertices on Q. This gives us a copy of $K_{t+1,q}$ in $G'\subseteq G$. ∎

Theorem 6.2. *Let k,ℓ,t,q be positive integers where $k\geq 4$. Let n be a sufficiently large positive integer depending on k,ℓ. There exists a constant C depending on k,ℓ,t,q such that every family $\mathcal{F}\subseteq\binom{[n]}{k}$ with $|\mathcal{F}|\geq t\binom{n}{k-1}+Cn^{k-2+\frac{2}{k-1}}$ contains either a copy of $\mathbb{P}_\ell^{(k)}$ or a copy of $[K_{t+1,q}]^{(k)}$.*

Proof. The set up of the proof will be similar to that in Section 5. Let $s=\max\{k\ell,kq(t+1)\}$. We may assume that $\mathcal{F}$ contains no copy of $\mathbb{P}_\ell^{(k)}$ and argue that $\mathcal{F}$ must contain $[K_{t+1,q}]^{(k)}$. Let $\mathcal{G}_1,\dots,\mathcal{G}_m,\mathcal{F}_0$ be a canonical partition of $\mathcal{F}$, where for each $i\in[m]$, $\mathcal{G}_i$ is (k,s)-homogeneous with intersection pattern $\mathcal{J}_i$ that has rank $k-1$ and is of type 1 and $|\mathcal{F}_0|\leq\frac{1}{c(k,s)}\binom{n}{k-2}$. Let $\mathcal{F}'=\bigcup_{i=1}^m\mathcal{G}_i$. Let H be the (k,s)-homogeneous graph of $\mathcal{F}'$ and H' the underlying undirected simple graph of H. Since $\mathcal{F}'$ doesn't contain $\mathbb{P}_\ell^{(k)}$, by Claim 1 of Lemma 4.4, H' has circumference less than ℓ and hence, $e(H')<\ell n$ and $e(H)\leq 2\ell n$. Therefore, $\sum_{x\in V(H)}d^+(x)\leq 2\ell n$. Let $D=n^{\frac{k-3}{k-1}}$. Define

$$A=\{x\in V(H)\colon d^+(x)\leq D\}$$
$$B=\{x\in V(H)\colon d^+(x)>D\}.$$

Let $\mathcal{F}_A$ denote the set of members F of $\mathcal{F}'$ whose central element $c(F)$ lies in A. We have

$$|\mathcal{F}_A|\leq|A|\cdot\binom{D}{k-1}<n\cdot D^{k-1}=n^{k-2}.$$

Since $\sum_{x\in V(H)} d^+(x) < 2\ell n$, we have $|B| < 2\ell n/D = 2\ell n^{\frac{2}{k-1}}$. The subgraph of H' induced by B, denoted by $H'[B]$, also has circumference at most 2ℓ and thus $e(H'[B]) < \ell|B| < 2\ell^2 n^{\frac{2}{k-1}}$. Let $\mathcal{F}_B$ denote the set of members of $\mathcal{F}'$ that contain edges of $H'[B]$. We have

$$|\mathcal{F}_B| \le e(H'[B])\binom{n-2}{k-2} < 2\ell^2 n^{k-2+\frac{2}{k-1}}.$$

Let $\widetilde{\mathcal{F}} = \mathcal{F}' \setminus (\mathcal{F}_A \cup \mathcal{F}_B) = \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_A \cup \mathcal{F}_B)$. By our discussions above, for large enough C we have

$$|\widetilde{\mathcal{F}}| \ge t\binom{n-1}{k-1} + \frac{C}{2} n^{k-2+\frac{2}{k-1}}. \tag{12}$$

By our definition of $\mathcal{F}_A$ and $\mathcal{F}_B$, we have

$$\widetilde{\mathcal{F}} = \{F \in \mathcal{F}' \colon c(F) \in B, |F \cap B| = 1\}.$$

Let $\widetilde{H}$ denote the subgraph of H consisting of all edges going from B to A. Note that $\widetilde{H}$ contains no multiple edges. We will ignore the directions on the edges of $\widetilde{H}$ and treat it as an undirected bipartite graph. Let $b = \binom{\ell}{t+1} \cdot q + \ell$. By Equation (12) and the proof of Lemma 4.3, we have

$$E(\widetilde{H}) > tn + Cn^{\frac{2}{k-1}} \ge t|A| + b|B|,$$

By Lemma 6.1, $\widetilde{H}$ contains either a copy of P_ℓ or a copy of $K_{t+1,q}$. Thus, $H \supseteq P_\ell$ or $H \supseteq K_{t+1,q}$. By Lemma 4.2, $\mathcal{F}$ contains either $\mathbb{P}_\ell^{(k)}$ or $[K_{t+1,q}]^{(k)}$. ∎

Theorem 6.2 yields

Corollary 6.3. *Let k, ℓ, t, q be positive integers where $k \ge 4$. We have*

$$\mathbf{ex}_k\left(n, \{\mathbb{P}_\ell^{(k)}, [K_{t+1,q}]^{(k)}\}\right) \le t\binom{n-1}{k-1} + O\left(n^{k-2+\frac{2}{k-1}}\right).$$

If we set $\ell = 2t+2$ and $q = t+2$, then since $K_{t+1,t+2} \supseteq P_{2t+2}$ Corollary 6.3 yields

Corollary 6.4. *Let k, t be positive integers where $k \ge 4$. For sufficiently large n we have*

$$\mathbf{ex}_k\left(n, \mathbb{P}_{2t+1}^{(k)}\right) \le \mathbf{ex}_k\left(n, \mathbb{P}_{2t+2}^{(k)}\right) \le t\binom{n-1}{k-1} + O\left(n^{k-2+\frac{2}{k-1}}\right).$$

This is almost as good as the bound in Theorem 4.5. However, Corollary 6.3 is a more general result, since ℓ and q can be an arbitrary constants independent of t. So with essentially the same bound, we get either the blow-up of a complete bipartite graph with $t+1$ vertices on one side or the blow-up of an arbitrarily long path.

7. Remarks and Problems

7.1. Triple systems

Theorem 2.4 (for linear paths) holds for $k \geq 4$. Our method does not quite work for the $k=3$ case. We **conjecture** that a similar result holds for $k=3$. On the other hand, as remarked at the end of Section 5, Theorem 2.5 (for loose paths) does hold for the $k=3$ case using the approach given in this paper.

7.2. Hamilton paths and cycles

Since the paper by Katona and Kierstead [19] there is a renewed interest concerning paths and (Hamilton) cycles in uniform hypergraphs. Most of these are Dirac type results (large minimum degree implies the existence of the desired substructure) like in Kühn and Osthus [23], Rödl, Ruciński, and Szemerédi [28] or in Dorbec, Gravier, and Sárközy [4].

7.3. Long paths

As the value of $c(k,s)$ in Lemma 3.1 is double exponentially small in k and s one can see that our exact results hold for

$$k\ell = O\left(\log\log n\right).$$

It would be interesting to close the gap, especially we **conjecture** that our result holds for much larger ℓ, maybe till $k\ell$ is as large as $O(n)$.

7.4. Linear trees

A family of sets $F_1,\dots,F_\ell$ is called a *linear* tree, if F_i meets $\cup_{j<i}F_j$ in exactly one vertex for all $1<i\leq\ell$. Any (usual, 2-uniform) tree $\mathbb{T}$ can be blown up in a natural way to a k-uniform linear tree $\mathbb{T}^{(k)}$. If the minimum number of vertices to cover all edges of $\mathbb{T}$ is τ, then $\mathbf{ex}_k(n,\mathbb{T}^{(k)}) \geq (\tau-1+o(1))\binom{n-1}{k-1}$. But one can make a better lower bound. Define

$$\sigma(\mathbb{T}) := \min\{|A| + e(\mathbb{T}\setminus A)\colon\, A\subset V(\mathbb{T})\,\text{independent}\}.$$

We have $\tau\leq\sigma\leq|V_1|\leq|V_2|$, where $V_1\cup V_2=V$ is the unique two-coloring of $\mathbb{T}$.

Define

$$\mathcal{F}_0^{(k)}(n,s) := \{F \in \binom{[n]}{k} : |F \cap \{1,2,\dots,s\}| = 1\}.$$

In case of $s<\sigma$ this hypergraph does not contain $\mathbb{T}^{(k)}$.

Theorem 7.1. [14] *Let $k\geq 3$. Let $\mathbb{T}$ be a tree. Let $\sigma=\sigma(\mathbb{T})$. Then*

$$(\sigma-1)\binom{n-\sigma+1}{k-1} \leq \mathbf{ex}\left(n,\mathbb{T}^{(k)}\right) \leq (\sigma-1+o(1))\binom{n-1}{k-1}.$$

It would be interesting to find asymptotics for the Turán numbers of other linear trees.

7.5. Kernel graphs and k-blow-ups

The Kernel graph approach we developed in this paper can potentially be very useful in attacking other hypergraph Turán problems, particularly the ones concerning k-blow-ups of other graphs besides paths. Some related notions of expanded graphs were investigated in earlier papers such as in [24], [26], and [29]. The use of appropriately defined auxiliary graphs may ultimately provide a useful approach for extending extremal results on graphs to hypergraphs.

7.6. The Erdős–Sós and the Kalai conjecture

A system of k-sets $\mathbb{T} := \{E_1, E_2, \dots, E_q\}$ is called a **tight** tree if for every $2\leq i\leq q$ we have $|E_i \setminus \cup_{j<i} E_j| = 1$, and there exists an $\alpha=\alpha(i)<i$ such that $|E_\alpha \cap E_i| = k-1$. The case $k=2$ corresponds to the usual trees in graphs. Let $\mathbb{T}$ be a k-tree on v vertices, and let $\mathbf{ex}_k(n,\mathbb{T})$ denote the maximum size of a k-family on n elements without $\mathbb{T}$. Consider a $P(n,v-1,k-1)$ packing $P_1, \dots, P_m$ on the vertex set $[n]$ (i.e., $|P_i| = v-1$ and $|P_i \cap P_j| < k-1$ for $1\leq i<j\leq m$) and replace each P_i by a complete k-graph. We obtain a $\mathbb{T}$-free hypergraph. Then Rödl's [27] theorem on almost optimal packings gives

$$\mathbf{ex}_k(n,\mathbb{T}) \geq (1-o(1))\frac{\binom{n}{k-1}}{\binom{v-1}{k-1}} \times \binom{v-1}{k} = (1-o(1))\frac{v-k}{k}\binom{n}{k-1}.$$

Conjecture 7.2. (Erdős and Sós for graphs, Kalai 1984 for all k, see in [10])

$$\mathbf{ex}_k\,(n,\mathbb{T}) \le \frac{v-k}{k}\binom{n}{k-1}.$$

The Erdős–Sós conjecture has been recently proved by a monumental work of Ajtai, Komlós, Simonovits, and Szemerédi [1], for $v \ge v_0$.

The Kalai conjecture has been proved for **star-shaped** trees in [10], i.e., whenever $\mathbb{T}$ contains a central edge which intersects all other edges in $k-1$ vertices. For $k=2$ these are the diameter 3 trees, 'brooms'.

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