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Linear Turán Numbers of Linear Cycles and Cycle-Complete Ramsey Numbers

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An r-uniform hypergraph is called an *r-graph*. A hypergraph is *linear* if every two edges intersect in at most one vertex. Given a linear r-graph H and a positive integer n, the *linear Turán number* $\mathrm{ex}_L(n,H)$ is the maximum number of edges in a linear r-graph G that does not contain H as a subgraph. For each $\ell \geqslant 3$, let C^r_ℓ denote the r-uniform linear cycle of length ℓ, which is an r-graph with edges $e_1,\dots,e_\ell$ such that, for all $i \in [\ell-1]$, $|e_i \cap e_{i+1}| = 1$, $|e_\ell \cap e_1| = 1$ and $e_i \cap e_j = \emptyset$ for all other pairs $\{i,j\}, i \neq j$. For all $r \geqslant 3$ and $\ell \geqslant 3$, we show that there exists a positive constant $c = c_{r,\ell}$, depending only r and ℓ, such that $\mathrm{ex}_L(n,C^r_\ell) \leqslant cn^{1+1/\lfloor \ell/2 \rfloor}$. This answers a question of Kostochka, Mubayi and Verstraëte [30]. For even ℓ, our result extends the result of Bondy and Simonovits [7] on the Turán numbers of even cycles to linear hypergraphs.

Using our results on linear Turán numbers, we also obtain bounds on the cycle-complete hypergraph Ramsey numbers. We show that there are positive constants $a = a_{m,r}$ and $b = b_{m,r}$, depending only on m and r, such that

$$R(C^r_{2m},K^r_t) \leqslant a\left(\frac{t}{\ln t}\right)^{m/(m-1)} \quad \text{and} \quad R(C^r_{2m+1},K^r_t) \leqslant bt^{m/(m-1)}.$$

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1. Introduction

We use standard notation and terminology. Notation and terminology that are specific to this paper are given in Section 2. A hypergraph H is *linear* if every pair of vertices in H is contained

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in at most one edge. Given a family $\mathcal{H}$ of r-graphs (that is, r-uniform hypergraphs), an r-graph G is said to be *H-free* if it does not contain any member of $\mathcal{H}$ as a subgraph, and the Turán number of H, denoted by $\mathrm{ex}(n,H)$, is the maximum number of edges in an n-vertex $\mathcal{H}$-free r-graph. In this paper, we consider the following variant of the Turán problem in linear hypergraphs.

Let $\mathcal{H}$ be a family of linear r-graphs. For each positive integer n, we define the *linear Turán number* of $\mathcal{H}$, denoted by $\mathrm{ex}_L(n,\mathcal{H})$, to be the maximum number of edges in an n-vertex $\mathcal{H}$-free linear r-graph. Note that in $\mathrm{ex}(n,\mathcal{H})$ the maximum is taken over all $\mathcal{H}$-free r-graphs on n vertices whereas in $\mathrm{ex}_L(n,\mathcal{H})$ the maximum is taken over all $\mathcal{H}$-free linear r-graphs on n vertices. So, for the same family $\mathcal{H}$ one can expect $\mathrm{ex}(n,\mathcal{H})$ and $\mathrm{ex}_L(n,\mathcal{H})$ to be very different. Indeed, if each member of $\mathcal{H}$ contains two disjoint edges, then $\mathrm{ex}(n,\mathcal{H}) \geqslant \binom{n-1}{r-1}$, whereas $\mathrm{ex}_L(n,\mathcal{H}) \leqslant O(n^2)$ always. When $\mathcal{H}$ consists of a single graph H, we write $\mathrm{ex}(n,H)$ and $\mathrm{ex}_L(n,H)$ for $\mathrm{ex}(n,\mathcal{H})$ and $\mathrm{ex}_L(n,\mathcal{H})$, respectively. In this paper, we study the linear Turán numbers of so-called r-uniform *linear cycles*.

A linear cycle of length ℓ is a hypergraph with edges $e_1,\dots,e_\ell$ such that, for all $i \in [\ell-1]$, $|e_i \cap e_{i+1}| = 1$, $|e_\ell \cap e_1| = 1$ and $e_i \cap e_j = \emptyset$ for all other pairs $\{i,j\}, i \neq j$. We denote an r-uniform linear cycle of length ℓ by C^r_ℓ. In particular, 2-uniform linear cycles are just the usual graph cycles. The Turán problem for 2-uniform cycles has been much studied. For odd cycles, the answer is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ for all sufficiently large n, with equality achieved by a balanced complete bipartite graph on n vertices. The problem for even cycles remains unresolved except for C_4 [18]. A general upper bound of $\mathrm{ex}(n,C_{2m}) \leqslant \gamma_m n^{1+1/m}$ for some positive constant γ_m was asserted by Erdős (unpublished). The first published proof was obtained by Bondy and Simonovits [7], who showed that $\mathrm{ex}(n,C_{2m}) \leqslant 20mn^{1+1/m}$ for all sufficiently large n. This was improved by Verstraëte [40] to $8(m-1)n^{1+1/m}$ and by Pikhurko [35] to $(m-1)n^{1+1/m}$. Very recently, Bukh and Jiang [9] improved the upper bound to $80\sqrt{m\log m}\cdot n^{1+1/m} + 10m^2 n$ for all $n \geqslant (2m)^{8m^2}$. For $m = 2,3,5$, constructions of C_{2m}-free n-vertex graphs with $\Omega(n^{1+1/m})$ edges are known (see [22]). Thus $\mathrm{ex}(n,C_{2m}) = \Theta(n^{1+1/m})$, for $m \in \{2,3,5\}$. However, the order of magnitude of $\mathrm{ex}(n,C_{2m})$ remains undetermined for all $m \notin \{2,3,5\}$.

The Turán problem for hypergraph cycles has also been explored. There are several different notions of hypergraph cycles. A hypergraph H is a *Berge cycle* of length ℓ if it consists of ℓ distinct edges $e_1,\dots,e_\ell$ such that there exists a list of distinct vertices $x_1,\dots,x_\ell$ satisfying that, for all $i \in [\ell-1]$, e_i contains both x_i and x_{i+1} and e_ℓ contains both x_ℓ and x_1. Note that a 2-uniform Berge cycle of length ℓ is just the usual graph cycle of length ℓ. For $r \geqslant 3$, however, r-uniform Berge cycles are not unique as there are no constraints on how the e_i intersect outside $\{x_1,\dots,x_\ell\}$. Let $\mathcal{B}^r_\ell$ denote the family of r-graphs that are Berge cycles of length ℓ. Győri and Lemons [24, 23] showed that for all $r \geqslant 3$, $\ell \geqslant 3$, there exists a positive constant $\beta_{r,\ell}$, depending on r and ℓ such that $\mathrm{ex}(n,\mathcal{B}^r_\ell) \leqslant \beta_{r,\ell} n^{1+1/\lfloor \ell/2 \rfloor}$. Another notion of hypergraph cycles that has been actively investigated recently is that of a linear cycle defined earlier. For fixed r,ℓ, the r-uniform linear cycle C^r_ℓ of length ℓ is unique up to isomorphism. We can also describe an r-uniform linear cycle using the notion of expansions. Given a 2-graph G, the *r-expansion* $G^{(r)}$ is the r-graph obtained from G by enlarging each edge of G into an r-set using $r-2$ new vertices, called *expansion vertices*, such that for different edges of G we use disjoint sets of expansion vertices. So an r-uniform linear cycle of length ℓ is precisely the r-expansion of a cycle of length ℓ. Füredi and Jiang [19] determined for all $r \geqslant 5$, $\ell \geqslant 3$ and sufficiently large n the exact value of $\mathrm{ex}(n,C^r_\ell)$, showing that

$$\mathrm{ex}(n, C^r_{2m+1}) = \binom{n}{r} - \binom{n-m}{r}, \quad \mathrm{ex}(n, C^r_{2m}) = \binom{n}{r} - \binom{n-m+1}{r} + \binom{n-m-1}{2},$$

respectively. Kostochka, Mubayi and Verstraëte [31] have subsequently showed that the same holds for all $r \geqslant 3$, $\ell \geqslant 3$ and sufficiently large n. In this paper, we study the linear Turán number of C^r_ℓ.

Determining $\mathrm{ex}_L(n, C^3_3)$ is equivalent to the famous $(6,3)$-problem, which is a special case of an old and general extremal problem of Brown, Erdős and Sós [8]. The Brown–Erdős–Sós problem asks to determine the function $f_r(n, v, e)$, which denotes the maximum number of edges in an r-graph on n vertices in which no v vertices span e or more edges. The problem of estimating $f_3(n,6,3)$ is known as the $(6,3)$-problem. It is easy to see that $\mathrm{ex}_L(n, C^3_3) = f_3(n,6,3)$ when n is sufficiently large. Additionally, as is well documented in the literature, $f_3(n,6,3)$ is closely related to the function $r_3(n)$, which denotes the largest size of a set of integers in $[n]$ not containing a 3-term arithmetic progression. Given n, let $m = \lfloor n/6 \rfloor$ and let A be a subset of size $r_3(m)$ that contains no 3-term arithmetic progression. Let X, Y, Z be disjoint sets with $X = [m], Y = [2m], Z = [3m]$, respectively. The 3-partite 3-graph

$$H = \{\{x,y,z\} : x \in X, y \in Y, z \in Z, \exists a \in A \; y = x + a, z = x + 2a\}$$

satisfies that no six points span three or more edges and $|H| = mr_3(m)$. Hence

$$f(n,6,3) \geqslant \left\lfloor \frac{n}{6} \right\rfloor \cdot r_3\left(\left\lfloor \frac{n}{6} \right\rfloor\right).$$

As one of the well-known applications of the regularity lemma, Ruzsa and Szemerédi [38] showed that $f_3(n,6,3) = o(n^2)$. This immediately implies Roth's theorem [36] that $r_3(n) = o(n)$. Since Roth's theorem [36] was established, the problem of estimating $r_3(n)$ has drawn much interest. The best current bounds are as follows: for some constant $c > 0$,

$$\frac{n}{e^{c\sqrt{\log n}}} \leqslant r_3(n) \leqslant \frac{n}{(\log n)^{1-o(1)}}. \tag{1.1}$$

Our discussions above yield the following bounds on $\mathrm{ex}_L(n, C^3_3)$.

Theorem 1.1. *For some constant $c > 0$,*

$$\frac{n^2}{e^{c\sqrt{\log n}}} < \mathrm{ex}_L(n, C^3_3) = o(n^2).$$

Using so-called 2-fold Sidon sets, Lazebnik and Verstraëte [32] constructed linear 3-graphs with girth 5 and $\Omega(n^{3/2})$ edges. On the other hand, it is not hard to show that $\mathrm{ex}_L(n, C^3_4) = O(n^{3/2})$. Hence $\mathrm{ex}_L(n, C^3_4) = \Theta(n^{3/2})$. Kostochka, Mubayi and Verstraëte [30] obtained the following bounds for $\mathrm{ex}_L(n, C^3_5)$.

Theorem 1.2 ([30]). *There are constants $a, b > 0$ such that $an^{3/2} < \mathrm{ex}_L(n, C^3_5) < bn^{3/2}$.*

No lower or upper bounds on $\mathrm{ex}_L(n, C_\ell^r)$, to our knowledge, have previously appeared in the literature for $\ell \notin \{3,4,5\}$. Kostochka, Mubayi and Verstraëte [30] asked if, for all $r \geqslant 3$, $\ell \geqslant 3$,

$$\mathrm{ex}_L(n, C_\ell^r) = O(n^{1+1/\lfloor \ell/2 \rfloor}).$$

We answer their question in the affirmative in our main theorem below.

Theorem 1.3 (main theorem). *For all $r, \ell \geqslant 3$, there exists a constant $c = c(r,\ell) > 0$, depending on r and ℓ, such that*

$$\mathrm{ex}_L(n, C_\ell^r) \leqslant cn^{1+1/\lfloor \ell/2 \rfloor}.$$

Another motivation for our study of $\mathrm{ex}_L(n, C_\ell^r)$ comes from the study of the hypergraph Ramsey number $R(C_\ell^r, K_t^r)$ of a linear cycle versus a complete graph. Such a study was initiated by Kostochka, Mubayi and Verstraëte in [29]. Using Theorem 1.3 and other tools, we obtain nontrivial upper bounds on $R(C_\ell^r, K_t^r)$. Since our main emphasis in the paper is on the linear Turán problem of linear cycles, we delay the discussion of the related Ramsey numbers to Section 7.

The rest of the paper is organized as follows. Section 2 contains some notation and terminology. Section 3 contains some lemmas needed for our main theorem. Section 4 contains the proof of the main theorem for even cycles. Section 5 contains some additional tools needed for the proof for odd cycles. Section 6 contains the proof of the main theorem for odd cycles (which is much more involved than for even cycles). Section 7 contains results on cycle-complete hypergraph Ramsey numbers. Section 8 contains concluding remarks. Our main method has roots in [16] and [25], but requires a substantial innovation for the odd cycle case. The new ideas used there may well have applications to other problems.

2. Notation and terminology

2.1. Degrees, neighbourhoods, link graphs

Let G be a hypergraph. Given a set $S \subseteq V(G)$, we define the *degree* of S in G, denoted by $d_G(S)$, to be the number of edges of G that contain S. Given a vertex $x \in V(G)$, we define the *link graph* $\mathcal{L}_G(x)$ of x in $\mathcal{G}$ as

$$\mathcal{L}_G(x) = \{e \setminus \{x\} : x \in e \in G\}.$$

Hence, if G is an r-graph, then $\mathcal{L}_G(x)$ is an $(r-1)$-graph. The *neighbourhood* $N_G(x)$ of x in G is defined as

$$N_G(x) = \{u \in V(G) : d_G(\{u,x\}) \geqslant 1\}.$$

When the context is clear, we will drop the subscripts in the above definitions.

2.2. r-expansions

Let k, r be integers where $r > k \geqslant 2$. Given a k-graph H, the *r-expansion* of H, denoted by $H^{(r)}$, is the r-graph obtained from H enlarging each edge e of H into an r-set through a set A_e of $r-k$ new vertices, called *expansion vertices*, such that whenever $e \neq e'$ we have $A_e \cap A_{e'} = \emptyset$. So, for instance, the r-expansion of a 2-uniform ℓ-cycle is precisely an r-uniform linear ℓ-cycle. We will call H the *skeleton* of $H^{(r)}$.

2.3. Levelled linear trees

Given a 2-uniform tree T rooted at w, for all $i \geqslant 0$, let $L_i = \{x : \mathrm{dist}_T(w,x) = i\}$. We call L_i *level* i of T. The *height* of T is the maximum i for which $L_i \neq \emptyset$. For each $x \in V(T)$, let T_x denote the subtree of T under x. Let $H = T^{(r)}$. Let f be a specific mapping of T to H that maps each $e \in T$ to $e \cup A(e)$ where $A(e)$ is the set of expansion vertices for e. We call H a *levelled linear r-tree* rooted at w and will refer to the L_i also as *levels* of H. We also refer to T as the *skeleton* of H. The height of H is defined to be the height of T. If x is a vertex in L_i for some i, then the *subtree under* x in H, denoted by H_x, is the image under f of T_x in H.

2.4. Proper, rainbow, strongly proper, strongly rainbow edge-colourings

Let c be an edge-colouring of a 2-graph G using natural numbers. We say that c is *proper* if $c(e) \neq c(e')$ whenever e and e' are incident edges in G, and we say that c is *rainbow* if we have $c(e) \neq c(e')$ for every two different edges e and e' in G. Let ϕ be an edge-colouring of a 2-graph G using p-subsets of some ground set S. We say that ϕ is *strongly proper* if $c(e) \cap c(e') = \emptyset$ whenever e and e' are incident edges in G. We say that ϕ is *strongly rainbow* if we have $c(e) \cap c(e') = \emptyset$ for every two different edges e and e' in G.

2.5. Default edge-colourings

Let G be an r-graph. The 2-shadow $\partial_2(G)$ of G is the 2-graph consisting of all pairs (a,b) that are contained in some edge of G. If G is linear then each edge in $\partial_2(G)$ is contained in a unique edge of G. We define the *default edge-colouring* ϕ of $\partial_2(G)$ by letting $\phi(\{a,b\}) = e \setminus \{a,b\}$, where e is the unique edge of G containing $\{a,b\}$. So ϕ is a colouring whose colours are $(r-2)$-sets. If $B \subseteq \partial_2(G)$ then the default edge-colouring of B is defined to be ϕ restricted to B.

3. Lemmas

In this section we prove some lemmas that will be needed in our main proofs in Sections 4 and 6. Let H be a hypergraph. A *vertex cover* of H is a set Q of vertices in H that contains at least one vertex of each edge of H. A *cross-cut* of H is a set S of vertices in H that contains exactly one vertex of each edge of H. A *matching* in H is a set of pairwise disjoint edges. The *size* of a matching is the number of edges in it.

Lemma 3.1. *Let H be a k-graph, where $k \geqslant 2$. Let Q be a minimum vertex cover of H. Then H contains a matching of size at least $|Q|/k$.*

Proof. Let M be a maximum matching in H and let S be the set of vertices contained in edges of M. If some edge e of H contains no vertex in S then $M \cup e$ is a larger matching in H than M, contradicting our choice of M. So S is a vertex cover of H of size $k|M|$. Since Q is a minimum vertex cover of H, we have $k|M| \geqslant |Q|$. Thus, $|M| \geqslant |Q|/k$. □

Lemma 3.2. *Let H be a k-graph, where $k \geqslant 2$. Let S be a vertex cover of H. Then there exist a subgraph $H' \subseteq H$ and a subset $S' \subseteq S$ such that $|H'| \geqslant (k/2^k)|H|$ and that S' is a cross-cut of H'.*

Proof. Let $\widetilde{S}$ be a random subset of S with each vertex of S chosen independently with probability $1/2$. For each $e \in H$, the probability that exactly one vertex of $e \cap S$ is included in $\widetilde{S}$ is

$$\frac{|e \cap S|}{2^{|e \cap S|}} \geqslant \frac{k}{2^k}.$$

So the expected number of edges e that intersects $\widetilde{S}$ at exactly one vertex is at least $(k/2^k)|H|$. Thus, there exists a subset $\widetilde{S}$ of S such that at least $(k/2^k)|H|$ edges intersect $\widetilde{S}$ at exactly one vertex. Let H' denote the subgraph of H consisting of these edges and $S' = \widetilde{S} \cap V(H')$. The claim follows. □

Lemma 3.3. *Let $r \geqslant 3$. Let G be a linear r-graph. Let $B \subseteq \partial_2(G)$ satisfy that each edge of G contains at most one edge of B. Let ϕ be the default edge-colouring of B. Then ϕ is strongly proper.*

Proof. Let f_1, f_2 be two edges in B that share a vertex, say u. Let e_1, e_2 be the unique edges of G containing f_1, f_2 respectively. By our assumption, $e_1 \neq e_2$. If $e_1 \setminus f_1$ and $e_2 \setminus f_2$ share a vertex v, then e_1, e_2 both contain $\{u, v\}$, contradicting G being linear. Thus $\phi(\{a,b\}) \cap \phi(\{a,c\}) = \emptyset$. □

Lemma 3.4. *Let k, ℓ, s be positive integers, where $k \geqslant 2$. Let G be a 2-graph with minimum degree at least $(k+1)\ell + s$. Let ϕ be a strongly proper edge-colouring of G using k-subsets of some set S. Let $x \in V(G)$ and $S_0 \subseteq S$ with $|S_0| \leqslant s$. Then there exists a path P in G of length ℓ starting at x such that* (i) *P is strongly rainbow under ϕ, and* (ii) *$\phi(f) \cap S_0 = \emptyset$ for all $f \in V(P)$.*

Proof. We use induction on ℓ. For the basis step, let $\ell = 1$. By our assumption, there are at least $k+s+1$ edges of G incident to x. Since ϕ is strongly proper, the colours used on these edges are pairwise disjoint k-sets. Certainly one of them is completely disjoint from S_0. Let e be an edge incident to x with $\phi(e) \cap S_0 = \emptyset$. The claim holds with $P = e$. For the induction step, let $\ell > 1$. By the induction hypothesis, there is a path P of length $\ell - 1$ starting at x such that (i) P is strongly rainbow under ϕ, and (ii) $\phi(f) \cap S_0 = \emptyset$ for all $f \in P$. Let $S_1 = \bigcup_{f \in P} \phi(f)$. Then $|S_1| = k(\ell - 1)$. Let y denote the other endpoint of P. There are at least $(k+1)\ell + s$ edges incident to y. More than $k\ell + s$ of these join y to vertices outside P. Since ϕ is strongly proper, the colours on these edges are pairwise disjoint k-subsets of S. Since $k\ell + s > k(\ell - 1) + s = |S_0 \cup S_1|$, for one of these edges e, we have $\phi(e) \cap (S_0 \cup S_1) = \emptyset$. Now, $P \cup e$ is a path of length ℓ in G starting at x such that (i) $P \cup e$ is strongly rainbow under ϕ, and (ii) $\phi(f) \cap S_0 = \emptyset$ for all $f \in P \cup e$. □

Lemma 3.5. *Let G be a graph with average degree d. There exists a subgraph $G' \subseteq G$ such that $\delta(G') \geqslant d/4$ and that $|G'| \geqslant |G|/2$.*

Proof. Suppose G has n vertices. Iteratively remove a vertex (and its incident edges) whose degree in the remaining subgraph is less than $d/4$ until no such vertex exists. Let G' denote the remaining subgraph. In the process, fewer than $nd/4 \leqslant \frac{1}{2}|G|$ edges have been removed. So $|G'| \geqslant |G|/2$. In particular, G' is non-empty. By our rule, we also have $\delta(G') \geqslant d/4$. $\square$

Lemma 3.6. *Let G be an r-graph with average degree d. Then G contains a subgraph G' with $\delta(G') \geqslant d/r$.*

Proof. Suppose G has n vertices. Starting with G, whenever some vertex has at most d/r in the remaining graph, we remove this vertex and all the edges in the remaining graph that contains this vertex. We repeat this procedure until there is no such vertex left. Let G' denote the remaining graph. Clearly, by our procedure at most $(n-1)(d/r) < nd/r = e$ edges have been removed in the process. So G' is non-empty. Also, by our condition, $\delta(G') \geqslant d/r$. $\square$

Below we give a version of the Chernoff bound from [34].

Lemma 3.7 (Chernoff bound). *Let X be the sum of n independent random variables $X_1,\dots,X_n$, where for each $i \in [n]$, $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1-p$. Then, for any real $0 \leqslant \alpha \leqslant 1$,*

$$\mathbb{P}(|X-np| > \alpha np) < 2e^{-(\alpha^2/3)np}.$$

Recall that given a hypergraph G and a vertex x, the *link graph* $L_G(x)$ of x in G is the graph $\{e \setminus \{x\}, e \in G, x \in e\}$. Given a set S of vertices in G, the *subgraph $G[S]$ of G induced by S* is the graph with vertex set S and edge set $\{e : e \in G, e \subseteq S\}$.

Proposition 3.8. *Let $c > 0$ be a fixed real. Let $m,r,t \geqslant 2$ be fixed positive integers. There exists a positive integer n_0 depending on c,m,r,t such that for all $n \geqslant n_0$ the following holds. Let G be a linear r-graph with $\delta(G) \geqslant cn^{1/m}$. Then there exists a partition of $V(G)$ into t sets $S_1,\dots,S_t$ such that, for each $u \in V(G)$ and each $i \in [t]$,*

$$|L_G(u) \cap G[S_i]| \geqslant \frac{c}{2t^{r-1}} n^{1/m}.$$

Proof. Independently and uniformly at random, assign each vertex in G a colour from $[t]$. For each $i \in [t]$ let S_i be the set of vertices receiving colour i. For each $u \in V(G), i \in [t]$, let $Y_{u,i}$ be the random variable that counts the number of edges in $L_G(u)$ completely contained in S_i. For fixed u,i, clearly each edge of $L_G(u)$ has probability $1/t^{r-1}$ of being contained in S_i. Since G is a linear r-graph, the edges of $L_G(u)$ are pairwise vertex-disjoint. So $Y_{u,i}$ is the sum of $d(u)$ independent random variables, each of which equals 1 with probability $p = 1/t^{r-1}$ and 0 with probability $1-p$. By Lemma 3.7,

$$\mathbb{P}\left(Y_{u,i} < \frac{1}{2}\frac{d(u)}{t^{r-1}}\right) < \mathbb{P}\left(\left|Y_{u,i} - \frac{d(u)}{t^{r-1}}\right| > \frac{1}{2}\frac{d(u)}{t^{r-1}}\right) < 2\exp\left(-\frac{1}{12}\frac{d(u)}{t^{r-1}}\right).$$

Since $d(u) \geqslant cn^{1/m}$, this yields

$$\mathbb{P}\left(Y_{u,i} < \frac{cn^{1/m}}{2t^{r-1}}\right) < 2\exp\left(-\frac{cn^{1/m}}{12t^{r-1}}\right).$$

Thus,

$$\mathbb{P}\left(\exists u \in V(G), \exists i \in [t], Y_{u,i} < \frac{cn^{1/m}}{2t^{r-1}}\right) < 2tn \cdot \exp\left(-\frac{cn^{1/m}}{12t^{r-1}}\right) < 1,$$

for all $n \geqslant n_0$, where n_0 depends only on c, m, r and t. Thus there exists a particular colouring for which

$$Y_{u,i} \geqslant \frac{cn^{1/m}}{2m^{r-1}} \quad \text{for all } u \in V(G),\ i \in [t].$$

Let $S_1, \ldots, S_t$ be the colour classes of this colouring. Then $(S_1, \ldots, S_t)$ forms a desired partition. $\square$

4. Linear Turán numbers of r-uniform even cycles

The following lemma provides the main ingredient of our proof of Theorem 1.3 for even cycles.

Lemma 4.1. *Let r, m, h be fixed integers, where $r \geqslant 3, m \geqslant 2, 0 \leqslant h \leqslant m-1$. For each positive integer i, let $c_i = 1/((rm2^{r+2})^i)$. Let G be a linear r-graph such that $C_{2m}^r \nsubseteq G$. Let H be an r-uniform levelled linear tree of height h rooted at w that is contained in G. Let $L_0, \ldots, L_h$ denote the levels of H. Let E be a set of edges in G each of which contains one vertex in L_h and $r-1$ vertices outside H. Suppose that $|E| \geqslant (m2^{r+3})^h |L_h|$. Then there exists a subset E^* of E such that $|E^*| \geqslant c_h |E|$ and that $E^* \setminus L_h$ is a matching. In particular, $H \cup E^*$ is a levelled linear tree of height $h+1$ rooted at w, with L_{h+1} consisting of one vertex of $e \setminus L_h$ for each $e \in E^*$.*

Proof. We use induction on h. For the basis step let $h = 0$, and let H consist of a single vertex w. By our assumption, E is a set of edges containing w. Since G is linear, every two of these edges intersect only at w. Let $E^* = E$. It is easy to see that the claim holds.

For the induction step, let $h \geqslant 1$. Suppose T is a 2-uniform tree of height h rooted at w with levels $L_0, L_1, \ldots, L_h$ and $H = T^{(r)} \subseteq G$. By our assumption, each edge in E contains one vertex in L_h and $r-1$ vertices outside H. Let $F = \{e \setminus L_h : e \in E\}$. Then F is an $(r-1)$-graph. Since G is linear and $r \geqslant 3$, the mapping $\sigma : E \to F$ that maps e to $e \setminus L_1$ is a bijection. So $|F| = |E|$. Let Q be a minimum vertex cover of F. By Lemma 3.2, there exist $F' \subseteq F$ and $Q' \subseteq Q$ such that

$$|F'| \geqslant \frac{r-1}{2^{r-1}}|F| = \frac{r-1}{2^{r-1}}|E|$$

and Q' is a cross-cut of F'. Let E' be the set of edges of E corresponding to edges of F' (via σ^{-1}). Then $|E'| = |F'|$ and each edge of E' contains exactly one vertex of L_h, one vertex of Q', and $r-2$ vertices outside $V(H) \cup Q'$. Let

$$B = \{e \cap (L_h \cup Q') : e \in E'\}.$$

By definition, B is a bipartite 2-graph with a bipartition (X, Q') where $X = V(B) \cap L_h$. The mapping $f : e \to e \cap (L_h \cup Q')$ is a bijection from E' to $B \subseteq \partial_2(G)$. So

$$|B| = |E'| = |F'| \geqslant \frac{r-1}{2^{r-1}}|E|.$$

Clearly, no edge of G contains more than one edge of B, and in the default edge-colouring ϕ of B the colours are disjoint from $V(B)\cup V(H)$.

Let $x_1,\dots,x_p$ denote the children of w in T. For each $i\in[p]$, let $A_i=V(T_{x_i})\cap L_h$. So A_i consists of vertices in L_h that are descendants of x_i (in T). Note that $A_1,\dots,A_p$ are pairwise disjoint. Let

$$Q^+=\{x\in Q':N_B(x)\cap A_i\neq\emptyset \text{ for at least } 2rm \text{ different } A_i\},$$
$$Q^-=\{x\in Q':N_B(x)\cap A_i\neq\emptyset \text{ for fewer than } 2rm \text{ different } A_i\}.$$

Then Q^+ and Q^- partition Q'. Let B^+ denote the subgraph of B induced by $X\cup Q^+$ and B^- the subgraph of B induced by $X\cup Q^-$.

Claim 4.2. *B^+ has average degree less than $4rm$.*

Proof of Claim 4.2. Suppose for contradiction that B^+ has average degree at least $4rm$. Then B^+ contains a subgraph B^* with minimum degree at least $2rm$. Let ϕ be the default edge-colouring of B^*. By Lemma 3.3, ϕ is strongly proper. Let x be any vertex in $V(B^*)\cap Q^+$. By Lemma 3.4, B^* contains a path P of length $2m-2h-2$ starting at x that is strongly rainbow under ϕ. Since B^* is bipartite and $2m-2h-2$ is even, the other endpoint y of P lies in Q^+. Now the r-graph P^+ with edge set $\{e\cup\phi(e):e\in P\}$ is a linear path of length $2m-2h-2$ with endpoints x and y using edges of $E'\subseteq E$. By the definition of E, $V(P^+)\cap V(H)\subseteq L_h$.

Since $x,y\in Q^+$, each sends edges in E' to at least $2rm$ different A_ℓ. We can find $i\neq j$ and $x'\in A_i, y'\in A_j$ and an edge $e_x\in E'$ containing x,x' and an edge $e_y\in E'$ containing y,y' such that $P^*=e_x\cup P^+\cup e_y$ is a linear path of length $2m-2h$ with endpoints x',y'. Indeed, since G is linear, the link of x in E' is a matching of size at least $2rm$, so we can find x' and e_x easily to extend P^+. Suppose $x'\in A_i$. We can find y' and e_y similarly with the additional requirement that $y'\in A_j$ for some $j\neq i$. Now, let P_x be the unique (x',w)-path and P_y the unique (y',w)-path in H, respectively. Since x',y' lie in different A_ℓ, P_x,P_y are two internally disjoint paths of length h, sharing only w. Now $P^*\cup P_1\cup P_2$ is a linear cycle of length $2m$ in G, contradicting our assumption about G. □

Claim 4.3.

$$|Q'|\geqslant\frac{c_{h-1}|B|}{8rm}.$$

Proof of Claim 4.3. By Claim 4.2, $|Q^+|\geqslant|B^+|/(4rm)$. If $|B^+|\geqslant\frac{1}{2}|B|$ then this yields

$$|Q'|\geqslant|Q^+|\geqslant\frac{|B|}{8rm}\geqslant\frac{c_{h-1}|B|}{8rm}$$

and we are done. Hence, we may assume that

$$|B^-|\geqslant\frac{|B|}{2}\geqslant\frac{(r-1)|E|}{2^r}.$$

For each vertex $x\in B^-$, by our assumption, $N_B(x)\cap A_i\neq\emptyset$ for fewer than $2rm$ different i. Among the A_i that receive edges of B^- from x, let $A_{i(x)}$ be one that receives the most edges of B from x. We now form a subgraph B_1^- of B^- by including for each $x\in Q^-$ the edges from x to $A_{i(x)}$. By our procedure,

$$|B_1^-| \geqslant \frac{|B^-|}{2rm} \geqslant \frac{(r-1)|E|}{rm2^{r+1}} \geqslant \frac{r-1}{rm2^{r+1}}(m2^{r+3})^h|L_h| \geqslant 2(m2^{r+3})^{h-1}|L_h|. \tag{4.1}$$

Recall that $A_1,\ldots,A_p$ are disjoint subsets of L_h. In B_1^-, each vertex in Q^- sends edges to at most one A_i. For each A_i, call A_i *light* if the number of edges of B_1^- incident to A_i is less than $(m2^{r+3})^{h-1}|A_i|$; otherwise call A_i *heavy*. Clearly the total number of edges of B^- that are incident to light A_i is at most $(m2^{r+3})^{h-1}|L_h|$, which is at most $\frac{1}{2}|B_1^-|$ by (4.1). So the number of edges of B_1^- that are incident to heavy A_i is at least $\frac{1}{2}|B_1^-|$.

Without loss of generality, suppose that $A_1,\ldots,A_t$ are the heavy A_i. For each $i\in[t]$, let Q_i be the set of vertices in Q^- that are joined by edges of B_1^- to A_i. By our definition, $Q_1,\ldots,Q_t$ are pairwise disjoint. For each $i\in[t]$, let E_i be the set of edges of E' corresponding to the set of edges of B_1^- that are incident to A_i. By our assumption, $|E_i| \geqslant (m2^{r+3})^{h-1}|A_i|$. Recall that $x_1,\ldots,x_p$ denote the children of w in T. For each $i\in[t]$, H_{x_i} is a linear tree of height $h-1$ rooted at x_i whose $(h-1)$th level is A_i. Each edge of E_i contains one vertex of A_i and $r-1$ vertices outside H_{x_i} and $|E_i| \geqslant (m2^{r+3})^{h-1}|A_i|$. By the induction hypothesis, there exists $E_i' \subseteq E_i$ such that $|E_i'| \geqslant c_{h-1}|E_i|$ and $E_i' \setminus A_i$ is a matching. In particular, this yields $|Q_i| \geqslant c_{h-1}|E_i|$. Hence, using (4.1), we have

$$|Q'| \geqslant |Q^-| \geqslant \sum_{i=1}^t |Q_i| \geqslant c_{h-1}\sum_{i=1}^t |E_i| \geqslant c_{h-1}\frac{|B_1^-|}{2} \geqslant \frac{c_{h-1}|B^-|}{4rm} \geqslant \frac{c_{h-1}|B|}{8rm}.$$

□

By Claim 4.3, we have

$$|Q| \geqslant |Q'| \geqslant \frac{c_{h-1}|B|}{8rm} \geqslant \frac{(r-1)c_{h-1}|E|}{2^{r-1}8rm} = \frac{(r-1)c_{h-1}|E|}{rm2^{r+2}}.$$

By Lemma 3.1, F contains a matching F^* of size at least

$$\frac{c_{h-1}|E|}{rm2^{r+2}} = c_h|E|.$$

Let E^* be the set of edges of E corresponding to F^*. Then $|E^*| = |F^*|$ and $H \cup E^*$ is a levelled linear tree of height $h+1$ rooted at w with L_{h+1} consisting of one vertex of each edges in F^*. □

Theorem 4.4. *Let m,r be positive integers where $m \geqslant 2$ and $r \geqslant 3$. There exist a positive real $c_{m,r}$ and a positive integer n_1 such that for all $n \geqslant n_1$ we have*

$$\mathrm{ex}_L(n, C_{2m}^r) \leqslant c_{m,r} n^{1+1/m}.$$

Proof. Let $\beta = (rm2^{r+2})^m$ and $c_{m,r} = 2m^{r-1}\beta$. Choose n_1 such that $c_{m,r}n_1^{1/m} \geqslant n_0$, where n_0 is given in Lemma 3.8. Let G be an n-vertex linear r-graph with at least $c_{m,r}n^{1+1/m}$ edges, where $n \geqslant n_1$. We prove that G contains a copy of C_{2m}^r. By our assumption, G has average degree at least $rc_{m,r}n^{1/m}$. By Lemma 3.6, there exists a subgraph G' of G with $\delta(G') \geqslant c_{m,r}n^{1/m}$. Let $N = n(G')$. Then $N \geqslant c_{m,r}n^{1/m} \geqslant n_0$ and $\delta(G') \geqslant c_{m,r}N^{1/m}$. By Lemma 3.8 (with $t = m$), there exists a partition of $V(G')$ into $S_1,\dots,S_m$ such that for each $u\in V(G')$ and $i\in[m]$, we have

$$|\mathcal{L}_{G'}(u)\cap G'[S_i]| \geqslant \frac{c_{m,r}}{2m^{r-1}}N^{1/m} = \beta N^{1/m}.$$

Let w be any vertex in S_1. Let $L_0=\{w\}$. Inside G', we will construct a levelled linear tree H of height m rooted at w with levels $L_1,\dots,L_m$ such that for each $i\in[m]$, $L_i\subseteq S_i$ and $|L_i|\geqslant N^{1/m}|L_{i-1}|$. This will imply that $|L_m|\geqslant N$, a contradiction, which will then complete our proof.

We construct H as follows. Let E_1 be the set of edges of G' containing w that correspond to $\mathcal{L}_{G'}(w)\cap G'[S_1]$. By our assumption, $|E_1|\geqslant \beta N^{1/m}\geqslant N^{1/m}$, by our definition of β. Also, each edge of E_1 consists of w and $r-1$ vertices in S_1. Let L_1 consist of a vertex from $e\setminus\{w\}$ for each $e\in E_1$. In general, suppose we have grown i levels $L_1,\dots,L_i$, where $i\leqslant m-1$, such that for each $j\in[i]$, $L_j\subseteq S_j$ and $|L_j|/|L_{j-1}|\geqslant N^{1/m}$. Let E_i denote the set of edges in G' that contain one vertex in L_i and $r-1$ vertices in S_{i+1}. By our assumption about the partition $(S_1,\dots,S_m)$, $|E_i|\geqslant \beta N^{1/m}|L_i|\geqslant (m2^{r+3})^m|L_i|$, noting that $\beta\geqslant (m2^{r+3})^m$. Since $C^r_{2m}\not\subseteq G'$, by Lemma 4.1, there exists a subset $E^*_i\subseteq E_i$ such that $|E^*_i|\geqslant (rm2^{r+2})^{-i}|E_i|$, for which $E^*_i\setminus L_i$ is a matching. Let $H_{i+1}=H_i\cup E^*_i$ and let L_{i+1} consist of one vertex from $e\setminus L_i$ for each $e\in E^*_i$. Then H_{i+1} is a levelled linear tree rooted at w of height $i+1$ whose $(i+1)$th level L_{i+1} is contained in S_{i+1}. Further,

$$|L_{i+1}| = |E^*_i| \geqslant \frac{1}{(rm2^{r+2})^i}|E_i| \geqslant \frac{\beta}{(rm2^{r+2})^i}N^{1/m}|L_i| \geqslant N^{1/m}|L_i|.$$

We continue this for m steps to obtain a levelled linear tree $H\subseteq G'$ with $|L_m|\geqslant N$, which yields the desired contradiction. □

5. Levelled linear quasi-trees

5.1. Levelled linear quasi-trees

To study the odd cycle case, we generalize the notion of levelled linear trees as follows. Let $r\geqslant 3$. A linear r-graph H is called a *levelled linear quasi-tree* of height h *rooted at* w if it is the union of a sequence of r-graphs $H_0,H_1,\dots,H_{h-1}$ satisfying the following.

(i) Each H_i is an r-partite r-graph with no isolated vertex and has parts $L_i,L'_i,J^{(1)}_i,\dots,J^{(r-2)}_i$ such that with $B_i=\{e\cap(L_i\cup L'_i): e\in H_i\}$, H_i is the r-expansion of B_i.

(ii) For each $i=0,1,\dots,h-1$, $J^{(r-2)}_i=L_{i+1}$.

(iii) For each $i=0,1,\dots,h-2$, $V(H_i)\cap V(H_{i+1})=L_{i+1}$ and $V(H_i)\cap V(H_j)=\emptyset$ whenever $|i-j|>1$.

(iv) $L_0=\{w\}$. For each $i=0,\dots,h$, we call L_i the ith *main level* of H. For each $i=0,\dots,h-1$, we call L'_i the ith *companion level* of H.

For each $i\in\{0,1,\dots,h-1\}$, we call H_i the ith *segment* of H and B_i *the defining bipartite graph* of H_i. For each edge f of B_i the unique vertex in L_{i+1} that corresponds to f is said to be a *representative* of e. Given $x\in V(B_i)$ and $y\in L_{i+1}$, we say that y is a *child* of x and that x is a *parent* of y if y is a representative of an edge of B_i incident to x. Observe that every two different vertices u,v in the same main level L_i or in the same companion level L'_i, where $i\leqslant h-1$, must have disjoint sets of children in L_{i+1} since the sets of edges of B_i incident to u and v, respectively, are disjoint.

Given a vertex $x \in L_i \cup L'_i$, where $i \leqslant h-1$, define the *down tree* T_x, rooted at x, to be the 2-graph obtained by including all the edges between $A_0 = \{x\}$ and its set A_1 of children in L_{i+1}, and then including all the edges joining vertices in A_1 and the set A_2 of their children in L_{i+2}, *etc.*, until we run out of levels. It is easy to see that T_x is a tree rooted at x of height at most $h-i$. Also, if $x,y \in L_i$ or $x,y \in L'_i$, $x \neq y$, then the earlier observation about disjoint sets of children implies that $V(T_x) \cap V(T_y) = \emptyset$. Furthermore, in T_w, where w is the root of H, for each $i = 0,\dots,h$, the ith distance class from w is precisely all of L_i.

Given a vertex $x \in L_i \cup L'_i$, where $i \leqslant h-1$, define the *down graph* H_x, rooted at x, to be the subgraph of H obtained by replacing each edge f of T_x with the corresponding edge e of H that contains f. The following lemma follows immediately from the definitions and our discussions above.

Lemma 5.1. *Let H be an r-uniform levelled linear quasi-tree of height h rooted at w with segments $H_1,\dots,H_{h-1}$. Let $x \in L_i \cup L'_i$, where $0 \leqslant i \leqslant h-1$. Then H_x is a levelled quasi-tree of height at most $h-i$ rooted at x. Also, for all $a,b \in L_i \cup L'_i$, $a \neq b$, if either $a,b \in L_i$ or $a,b \in L'_i$, then $(V(H_a) \cap V(H_b)) \cap L_j = \emptyset$ for all $j \geqslant i+1$.*

In a linear r-graph, a path P is just the r-expansion of a 2-uniform path. An endpoint of P is a vertex in the first or last edge that has degree 1 in P. An (x,y)-path is a path where x is an endpoint in the first edge of P and y is an endpoint in the last edge of P (or *vice versa*).

Lemma 5.2. *Let H be an r-uniform levelled linear quasi-tree of height h rooted at w with segments $H_1,\dots,H_{h-1}$, where $L_0,L_1,\dots,L_h$ and $L'_0,\dots,L'_{h-1}$ denote the main levels and companion levels, respectively. Let $x,y \in L_i, x \neq y$, where $1 \leqslant i \leqslant h-1$. Then there exists an (x,y)-path P of even length at most $2i$ that is contained in $\bigcup_{j=0}^{i-1} H_j$ and intersects L_i only at x and y.*

Proof. We use induction on i. The claim is trivial when $i=1$. So assume $i \geqslant 2$. Let e be the unique edge of H_i that contains x and f the unique edge of H_i that contains y. If e and f share a vertex, then $e \cup f$ is an (x,y)-path of length 2. Otherwise $e \cap f = \emptyset$. Let $\{x'\} = e \cap L_{i-1}$ and $\{y'\} = f \cap L_{i-1}$. By the induction hypothesis, there is an (x',y')-path P of even length at most $2(i-1)$ that is contained in $\bigcup_{j=0}^{i-2} H_j$ and intersects L_{i-1} only at x' and y'. Now, $P \cup \{e,f\}$ is an (x,y)-path of even length at most $2i$ that is contained in $\bigcup_{j=0}^{i-1} H_j$ and intersects L_i only at x and y. $\square$

Given a levelled linear quasi-tree H rooted at w, a *monotone path* is a path in H that hits each main level at most once. It is easy to see that for every vertex x in the ith main level, there is a unique monotone (w,x)-path, and that path has length i. For every vertex y in the ith companion level, there exists at least one monotone (w,y)-path and such a path has length $i+1$.

An r-uniform *spider* F with t legs consists of t many r-uniform linear paths $P_1,\dots,P_t$ (called the *legs*) sharing one endpoint x but otherwise vertex-disjoint.

Lemma 5.3. *Let h,p,r be positive integers, where $r \geqslant 3$. Let H be an r-uniform levelled linear quasi-tree of height h rooted at w with segments $H_1,\dots,H_{h-1}$. Let $L_0,\dots,L_h$ and $L'_0,\dots,L'_{h-1}$ be* *the main levels and companion levels, respectively. Let $S \subseteq L_h$ such that $|S| \geqslant (hpr)^h$. Then there exists a vertex $x \in V(H)$ such that* (i) $|V(H_x) \cap S| \geqslant 1/((hpr)^{h-1})|S|$, *and* (ii) H_x *contains a spider centred at x that has p legs, each of which is a monotone path from x to $V(H_x) \cap S$.*

Proof. We use induction on h. For the basis step let $h = 1$. In this case, the claim clearly holds by choosing x to be w and p of the edges containing x to form the required spider. For the induction step, let $h \geqslant 2$. Clearly there is at least one monotone path from the root w to S, so there exist spiders centred at w with legs being monotone paths from w to S. Let us call these (w,S)-*spiders*. Among all (w,S)-spiders, let M be one that has the maximum number of legs. If M has p legs, then the claim holds with $x = w$. So assume M has fewer than p legs. For each $y \in S$, let P_y be the unique monotone path in H from w to y. The maximality of M implies that each $y \in S$, P_y intersects M somewhere other than w. Let $y \in S$. If P_y intersects M at a vertex u in $V(H_i) \setminus \{L_i, L_i'\}$ for some $i \leqslant h-1$, then such a vertex is an expansion vertex in H_i, and both P_y and M must contain the corresponding edge e of H_i that contains u and hence both contain $e \cap L_i$ and $e \cap L_{i+1}$. Thus, for each $y \in S$, P_y contains a vertex in

$$U = (V(F) \setminus \{w\}) \cap \left(\bigcup_{i=1}^{h-1} (L_i \cup L_i') \right).$$

Since U has fewer than phr vertices, by the pigeonhole principle, there exists a vertex z in U that is contained in at least $\lceil s/(hpr) \rceil$ different P_y. Suppose that $z \in L_a \cup L_a'$. Let S' be the set of vertices y in S such that P_y contains z. Then $|S'| \geqslant |S|/(hpr)$. For each $y \in S'$, let P_y' be the (z,y)-path contained in P_y. Let $H' = \bigcup_{y \in S} P_y'$. Then $H' \subseteq H_z$. Now, H_z is a levelled linear quasi-tree with height at most $h-1$ and S' is a set of vertices in its last level. By the induction hypothesis, there is a vertex x in H_z such that

$$|V((H_z)_x) \cap S'| \geqslant \frac{|S'|}{[(h-1)pr]^{h-2}} \geqslant \frac{|S|}{(hpr)^{h-1}}$$

and $(H_z)_x$ contains a $(z, V(H_z(x)) \cap S')$-spider with p legs. Consider now the relationship between $(H_z)_x$ and H_x. Since x sends multiple internally disjoint monotone paths to S, it is easy to see that either $x = z$ or $x \in L_j \cup L_j'$ for some $j \geqslant a+1$. In either case, we have $(H_z)_x = H_x$. □

6. Linear Turán numbers of odd cycles

The following lemma provides the key ingredient for our proof of Theorem 1.3 for odd cycles. Before presenting the technical details, let us point out the main technical challenge for the odd cycle case and the key new ideas for overcoming the difficulty. The general plan is similar to the even cycle case. We use a linear quasi-tree as a framework for growing levels and argue that in the absence of C_{2m+1}^r the graph must expand quickly. The main difficulty we face is that linear quasi-trees have an interweaving structure and no longer possess a clean tree structure. Therefore, we cannot hope to link vertices cleanly back to the root. The key idea in overcoming this difficulty is to apply Lemma 5.3 to locate a set of vertices (called ‘dominators’) at some earlier level to act as different roots for different vertices. This is the main innovation to the usual approach based on breadth-first search.

Lemma 6.1. *Let r,m,h be integers, where $r \geqslant 3$, $m \geqslant 2$ and $0 \leqslant h \leqslant m-1$. Let $\lambda = 2m^2r^2$ and $c = 2^{r+2}\lambda^m$. Let G be a linear r-graph such that $C^r_{2m+1} \not\subseteq G$. Let H be an r-uniform levelled linear quasi-tree of height h in G rooted at w with segments $H_0, H_1, \dots, H_{h-1}$, levels $L_0, L_1, \dots, L_h$ and companion levels $L'_1, \dots, L'_{h-1}$. Let E be a set of edges in G, each of which contains one vertex in L_h and $r-1$ vertices outside H. Suppose that $|E| \geqslant c^h|L_h|$. Then there exist a subset E^* of E and a set S of vertices outside H such that*

(i) $|E^*| \geqslant (1/c^h)|E|$,
(ii) *S is a cross-cut of E^*,*
(iii) *E^* is the r-expansion of the 2-graph $\Gamma = \{e \cap (L_h \cup S) : e \in E^*\}$, and*
(iv) *either $\delta(\Gamma) \geqslant 2mr$ or Γ is a disjoint union of stars with centres in L_h and leaves in S.*

In particular, $H \cup E^*$ is a levelled linear quasi-tree of height $h+1$ rooted at w, where $L'_h = S$ and L_{h+1} consists of one vertex from each member of $E^* \setminus (L_h \cup S)$.

Proof. We use induction on h. For the basis step, let $h = 0$. Then H consists of the single vertex w and E is a set of edges containing w. Let $E^* = E$ and let S consist of one vertex of $e \setminus \{w\}$ for each $e \in E^*$. It is easy to see that the claim holds.

For the induction step, let $h \geqslant 1$. Let E be defined as in the statement of the lemma. Let $F = \{e \setminus L_h : e \in E\}$. Then F is an $(r-1)$-graph with $|F| = |E|$. Let Q be a minimum vertex cover of F. If $|Q| \geqslant ((r-1)/c^h)|E|$, then by Lemma 3.1, F contains a matching F^* of size at least

$$\frac{|Q|}{r-1} \geqslant \frac{1}{c^h}|E|.$$

By letting E^* be the set of edges of E corresponding to F^* and letting $S = E^* \cap Q$, it is easy to check that E^* and S satisfy the four conditions and we are done. We henceforth assume that

$$|Q| < \frac{r-1}{c^h}|E|. \tag{6.1}$$

By Lemma 3.2, there exist $F' \subseteq F$ and $Q' \subseteq Q$ such that

$$|F'| \geqslant \frac{r-1}{2^{r-1}}|F|$$

and Q' is a cross-cut of F'. Let E' be the set of edges in E corresponding to F'. Then $|E'| = |F'|$ and each edge in E' intersects each of L_h and Q' at exactly one vertex. Let

$$B = \{e \cap (L_h \cup Q') : e \in E'\}.$$

Then B satisfies the condition of Lemma 3.3 and there is a bijection between edges of B and edges of E'. In particular,

$$|B| = |E'| \geqslant \frac{r-1}{2^{r-1}}|E|. \tag{6.2}$$

Now,

$$|B| \geqslant \frac{r-1}{2^{r-1}}|E| \geqslant \frac{(r-1)c^h}{2^{r-1}}|L_h| > \frac{c^h}{2^{r-1}}|L_h|.$$

Also, by (6.1) and (6.2),

$$|B| \geqslant \frac{r-1}{2^{r-1}}|E| \geqslant \frac{c^h}{2^{r-1}}|Q|.$$

So,

$$|B| \geqslant \frac{c^h}{2^r}(|L_h|+|Q|) \geqslant \frac{c^h}{2^r}|V(B)|.$$

Let $d(B)$ denote the average degree of B. Then

$$d(B) \geqslant \frac{c^h}{2^{r-1}} \geqslant 32hmr. \tag{6.3}$$

Recall that

$$\lambda = 2m^2r^2 \quad \text{and} \quad c = 2^{r+2}\lambda^m. \tag{6.4}$$

We now partition Q' as follows. Let

$$Q^- = \{y \in Q' : d_B(y) < \lambda^m\} \quad \text{and} \quad Q^+ = \{y \in Q' : d_B(y) \geqslant \lambda^m\}.$$

Let B^- be the subgraph of B induced by $L_h \cup Q^-$ and let B^+ be the subgraph of B induced by $L_h \cup Q^+$. Then, by (6.1) and (6.4),

$$|B^-| \leqslant \lambda^m|Q^-| \leqslant \lambda^m\frac{r-1}{c^h}|E| < \frac{r-1}{2^{r+2}}|E| < \frac{|B|}{2}. \tag{6.5}$$

Hence,

$$|B^+| > \frac{|B|}{2}. \tag{6.6}$$

From this point on, we will just work with B^+. We further partition Q^+ as follows. For convenience, let $L'_0 = L_0 = \{w\}$. Let $y \in Q^+$. Then

$$N_B(y) \subseteq L_h \quad \text{and} \quad |N_B(y)| \geqslant \lambda^m \geqslant \lambda^h \geqslant (2h^2r^2)^h.$$

By Lemma 5.3, there exists $x \in V(H)$ such that

$$|V(H_x) \cap N_B(y)| \geqslant \frac{|N_B(y)|}{\lambda^{h-1}}$$

and such that H_x contains a spider M with $2hr$ legs from x to $N_B(y)$ using monotone paths. Suppose $x \in L_i \cup L'_i$. By adding to M the $2hr$ edges of E' that contain y and intersect $V(M) \cap N_B(y)$, we obtain $2hr$ internally disjoint (x,y)-paths of length $h-i+1$ contained in $(\bigcup_{j=i}^{h-1} H_j) \cup E$. Let us call such an x a *dominator* of y in H. For each $y \in Q^+$, fix a dominator $\alpha(y)$ of y in H. Note that either $\alpha(y) = w$ or $\alpha(y) \in L_i \cup L'_i$ for some $1 \leqslant i \leqslant h-1$, since other vertices in H have degree 1 in H and cannot possibly be a dominator of y. Let us summarize properties of a dominator as follows.

Claim 6.2. *Let $y \in Q^+$. Then*

$$|V(H_{\alpha(y)}) \cap N_B(y)| \geqslant \frac{|N_B(y)|}{\lambda^{h-1}}.$$

If $\alpha(y) \in L_i \cup L_i'$, *then there exist* $2hr$ *internally disjoint* $(\alpha(y), y)$*-paths of length* $h-i+1$ *contained in* $(\bigcup_{j=i}^{h-1} H_j) \cup E$.

We now partition Q^+ according to which level the fixed dominator of a vertex lies in. Let

$$Q_0 = \{y \in Q^+ : \alpha(y) = w\}.$$

For each $i = 1, \ldots, h-1$, let

$$Q_i = \{y \in Q^+ : \alpha(y) \in L_i\} \quad \text{and} \quad Q_i' = \{y \in Q^+ : \alpha(y) \in L_i'\}.$$

Then $Q_0, Q_1, Q_1', \ldots, Q_{h-1}, Q_{h-1}'$ partition Q^+.

Let B_0 denote the subgraph of B^+ induced by $Q_0 \cup L_h$. For each $i = 1, \ldots, h-1$, let B_i denote the subgraph of B^+ induced by $Q_i \cup L_h$ and let B_i' denote the subgraph of B^+ induced by $Q_i' \cup L_h$. Then $B_0, B_1, B_1', \ldots, B_{h-1}, B_{h-1}'$ partition B^+. One of these graphs must then have size at least $|B^+|/2h$. We consider three cases, depending on which B_i has size at least $|B^+|/2h$. In each case, we will find a set E^* and a set S of vertices that satisfy the four conditions of the lemma. This will complete our proof.

Case 1. $|B_0| \geqslant |B^+|/2h$.

In this case, we have $|B_0| \geqslant |B|/4h$. Since B has average degree at least $32hmr$ by (6.3), B_0 has average at least $8mr$. By Lemma 3.5, B_0 contains a subgraph B_0' with

$$\delta(B_0') \geqslant 2mr \quad \text{and} \quad |B_0'| \geqslant \frac{|B_0|}{2} \geqslant \frac{|B|}{8h}. \tag{6.7}$$

Let E^* be the set of edges in E corresponding to those in B_0', let $S = V(B_0') \cap Q_0$ and $\Gamma = B_0'$. Then using (6.2) and (6.7), we have

$$|E^*| = |B_0'| \geqslant \frac{|B|}{8h} \geqslant \frac{r-1}{2^{r+2}h}|E| \geqslant \frac{1}{c^h}|E|. \tag{6.8}$$

So condition (i) of the lemma holds. Also, condition (ii) holds since $E^* \subseteq E'$, $Q_0 \subseteq Q'$ and Q' is a cross-cut of E'. Also, condition (iv) holds by (6.7). So, it remains to show that condition (iii) holds, that is, E^* is the r-expansion of Γ. This is equivalent to saying that the default colouring ϕ on B_0' is strongly rainbow. Suppose for contradiction that ϕ is not strongly rainbow. Then there exist $e, e' \in B_0'$ such that $e \neq e'$ and $\phi(e) \cap \phi(e') \neq \emptyset$. By Lemma 3.3, ϕ is strongly proper on B_0'. So, e, e' are two independent edges in B_0'. Let $v \in \phi(e) \cap \phi(e')$. Since H is linear, we have $\phi(e) \cap \phi(e') = \{v\}$. Suppose $e = xy, e' = x'y'$, where $x, x' \in L_h$ and $y, y' \in Q_0$. Since B_0' has minimum degree at least $2mr$, by Lemma 3.4, B_0' contains a path P of length $2m-2-2h$ starting at y' such that P is strongly rainbow under ϕ and

$$\left(\bigcup_{f \in P} \phi(f)\right) \cap (\phi(e) \cup \phi(e')) = \emptyset.$$

Let y'' denote the other endpoint of P. The set of edges of E that correspond to those in $P \cup \{e, e'\}$ forms a linear path R of length $2m-2h$ in which we may view x as one endpoint at one end and y'' as an endpoint at the other end. Let R_x be a monotone path in H from w to x. Then $R \cup R_x$ is a linear path of length $2m-2h+h = 2m-h$ with w being an endpoint at one end and y'' being an endpoint at the other end. Since $y'' \in Q_0$, w is a dominator of y''. By Claim 6.2. there exist $2hr$

pairwise internally disjoint (w,y'')-paths of length $h+1$. Since $2hr > |V(R\cup R_x)|$, one of these paths, say R', is internally disjoint from $R\cup R_x$. Now $R\cup R_x\cup R'$ is a linear cycle of length $2m+1$ in G, a contradiction. This completes Case 1.

Case 2. $|B_i| \geqslant |B^+|/2h$ for some $1\leqslant i\leqslant h-1$.

Fix such an i. For each $x\in L_i$, let $A_x = L_h\cap V(H_x)$. By Lemma 5.1,

$$A_x\cap A_{x'} = \emptyset, \quad \text{for all } x,x'\in L_i,\ x\neq x'. \tag{6.9}$$

Let $y\in Q_i$. By definition, $\alpha(y)\in L_i$. Thus, we have

$$|V(H_{\alpha(y)}\cap N_B(y)| \geqslant \frac{|N_B(y)|}{\lambda^{h-1}}.$$

This is equivalent to

$$|A_{\alpha(y)}\cap N_B(y)| \geqslant \frac{|N_B(y)|}{\lambda^{h-1}}.$$

We define a subgraph $\widetilde{B}_i$ of B_i by including only the edges of B_i from y to $A_{\alpha(y)}\cap N_B(y)$ for each $y\in Q_i$. Let $\widetilde{E}_i$ denote the set of edges in E corresponding to $\widetilde{B}_i$. Then $|\widetilde{E}_i| = |\widetilde{B}_i|$. Using (6.2) and $\lambda = 2m^2r^2$, $c = 2^{r+2}\lambda^m > 2^{r+3}h\lambda^{h-1}$, we have

$$|\widetilde{E}_i| = |\widetilde{B}_i| \geqslant \frac{|B_i|}{\lambda^{h-1}} \geqslant \frac{|B|}{4h\lambda^{h-1}} \geqslant \frac{r-1}{2^{r+1}h\lambda^{h-1}}|E| > \frac{4(r-1)}{c}|E|. \tag{6.10}$$

For each $x\in L_i$, let $\widetilde{B}_x$ denote the subgraph of $\widetilde{B}_i$ consisting of edges of $\widetilde{B}_i$ that are incident to A_x and let $C_x = V(\widetilde{B}_x)\cap Q_i$. So, for each $x\in L_i$, $\widetilde{B}_x$ is a bipartite graph with a bipartition (A_x,C_x). For each $x\in L_i$, let $\widetilde{E}_x$ denote the set of edges in E corresponding to $\widetilde{B}_x$. Then $|\widetilde{E}_x| = |\widetilde{B}_x|$ and both A_x and C_x are cross-cuts of $\widetilde{E}_x$. By our definition of $\widetilde{B}_i$, $\widetilde{B}_x$ and (6.9),

$$C_x\cap C_y = \emptyset, \quad \text{for all } x,y\in L_i,\ x\neq y. \tag{6.11}$$

Hence,

$$\widetilde{B}_x\cap\widetilde{B}_y = \emptyset \quad\text{and}\quad \widetilde{E}_x\cap\widetilde{E}_y = \emptyset, \quad \text{for all } x,y\in L_i,\ x\neq y. \tag{6.12}$$

We call x *light* if $|\widetilde{E}_x| = |\widetilde{B}_x| \leqslant c^{h-1}|A_x|$ and *heavy* if $|\widetilde{E}_x| = |\widetilde{B}_x| > c^{h-1}|A_x|$. Clearly, the combined size of $\widetilde{E}_x$ over all light x in L_i is at most $c^{h-1}|L_h| \leqslant \frac{1}{2}|\widetilde{E}_i|$, where the last inequality follows from (6.10) and the assumption that $|E| \geqslant c^h|L_h|$. Hence,

$$\sum_{x \text{ is heavy}} |\widetilde{E}_x| \geqslant \frac{1}{2}|\widetilde{E}_i|. \tag{6.13}$$

Now, consider any heavy $x\in L_i$. Since H_x is a levelled linear quasi-tree rooted at x of height $h-i\leqslant h-1$ with last level A_x and $\widetilde{E}_x$ is a set of at least $c^{h-1}|A_x|$ edges each of which contains one vertex in A_x and $r-1$ vertices outside H_x, we can apply the induction hypothesis to obtain $E^*_x\subseteq\widetilde{E}_x$ and S_x that satisfy the four conditions of the lemma. That is,

(i) $|E^*_x| \geqslant (1/c^{h-1})|\widetilde{E}_x|$,
(ii) S_x is a cross-cut of E^*_x outside H_x (by our definition of E, S_x is also outside H),
(iii) E^*_x is the r-expansion of $\Gamma_x = \{e\cap(A_x\cup S_x) : e\in E^*_x\}$, and

(iv) either (a) $\delta(\Gamma_x) \geqslant 2mr$ or (b) Γ_x is a disjoint union of stars with centres in A_x and leaves in S_x.

We say that x is of 'type 1' if (a) holds and that x is of 'type 2' if (b) holds in condition (iv). Let

$$L_{i,1} = \{x \in L_i, x \text{ is heavy and is of type } 1\},$$
$$L_{i,2} = \{x \in L_i, x \text{ is heavy and is of type } 2\}.$$

Note that if x is of type 2, then by case (b) of condition (iv), E^*_x consists of a collection of linear stars in which vertices outside A_x all have degree 1. Since C_x is a cross-cut of $\widetilde{E}_x$, it contains at least one vertex from each edge in E^*_x. Each such vertex is a degree 1 vertex in E^*_x. Hence, we have

$$|C_x| \geqslant |E^*_x|, \quad \text{for all } x \in L_{i,2}. \tag{6.14}$$

If $\sum_{x\in L_{i,2}} |\widetilde{E}_x| \geqslant \frac{1}{4}|\widetilde{E}_i|$, then by condition (i), (6.10) and (6.14) we have

$$|Q_i| \geqslant \sum_{x\in L_{i,2}} |C_x| \geqslant \sum_{x\in L_{i,2}} |E^*_x| \geqslant \frac{1}{c^{h-1}} \sum_{x\in L_{i,2}} |\widetilde{E}_x| \geqslant \frac{1}{4c^{h-1}}|\widetilde{E}_i| \geqslant \frac{1}{4c^{h-1}}\frac{4(r-1)}{c}|E| = \frac{r-1}{c^h}|E|, \tag{6.15}$$

contradicting (6.1). Hence, by (6.13), we may assume that

$$\sum_{x\in L_{i,1}} |\widetilde{E}_x| \geqslant \frac{1}{4}|\widetilde{E}_i|. \tag{6.16}$$

Let

$$E^* = \bigcup\{E^*_x : x \in L_{i,1}\}, \quad S = \bigcup\{S_x : x \in L_{i,1}\} \quad \text{and} \quad \Gamma = \bigcup\{\Gamma_x : x \in L_{i,1}\}.$$

We now verify that E^*, S and Γ satisfy the four conditions of the lemma, which would complete Case 2. Since

$$|E^*_x| \geqslant \frac{1}{c^{h-1}}|\widetilde{E}_x|$$

for each $x \in L_{i,1}$, using the last two inequalities in (6.15), we have

$$|E^*| \geqslant \frac{1}{c^{h-1}} \sum_{x\in L_{i,1}} |\widetilde{E}_x| \geqslant \frac{1}{4c^{h-1}}|\widetilde{E}_i| \geqslant \frac{r-1}{c^h}|E| \geqslant \frac{1}{c^h}|E|.$$

Hence condition (i) holds. By the definitions of E^* and S, condition (ii) holds. Since Γ_x are vertex-disjoint over different $x \in L_{i,1}$ and $\delta(\Gamma_x) \geqslant 2mr$ for each $x \in L_{i,1}$, we have $\delta(\Gamma) \geqslant 2mr$. Hence condition (iv) holds. It remains to verify that condition (iii) holds, that is, E^* is the r-expansion of Γ. In other words, we need to verify that the default edge-colouring ϕ of Γ is strongly rainbow.

By our assumption, for all $x \in L_{i,1}$, E^*_x is the r-expansion of Γ_x. Hence the default edge-colouring of Γ_x is strongly rainbow. Thus, it remains to show that whenever $x, y \in L_{i,1}, x \neq y$ and $e \in \Gamma_x, f \in \Gamma_y$ we have $\phi(e) \cap \phi(f) = \emptyset$. Let $x, y \in L_{i,1}$ such that $x \neq y$ and let $e \in \Gamma_x, f \in \Gamma_y$.

By Lemma 5.2, there exists an (x,y)-path R_0 of some even length $2j \leqslant 2i$ in $\bigcup_{t\leqslant i} H_t$ that intersects L_i only at x and y. Suppose for contradiction that $\phi(e) \cap \phi(f) \neq \emptyset$. Let $v \in \phi(e) \cap \phi(f)$. Since H is linear, we have $\phi(e) \cap \phi(f) = \{v\}$. Suppose $e = ab$ and $f = a'b'$, where $a \in A_x, b \in C_x$

and $a' \in A_y, b' \in C_y$. Let

$$\ell = 2m - [2j + 2(h-i) + 2] = 2m - 2 - 2h + 2(i-j).$$

Note that ℓ is even and satisfies $2m - 2 - 2h \leqslant \ell \leqslant 2m-4$. Since $\delta(\Gamma_y) \geqslant 2mr > r\ell + 2r$, by Lemma 3.4, there exists a path P in Γ_y of length ℓ starting at b' that is strongly rainbow under ϕ and such that

$$\left(\bigcup_{e' \in P} \phi(e')\right) \cap (\phi(e) \cup \phi(f)) = \emptyset.$$

Let b'' denote the other endpoint of P. Since P has even length, $b'' \in C_y$. Let P^+ denote the set of the edges of E that correspond to the edges of $P \cup \{e, f\}$. Then P^+ is linear path of length $2m - 2h + 2(i-j)$ in G where a is an endpoint at one end and b'' is an endpoint at the other end. Furthermore, $V(P^+) \cap V(H) \subseteq L_h$. Let R be a monotone path in H from x to a. Then R has length $h-i$ and is internally disjoint from R_0 and P^+. Since $b'' \in C_y$, y is a dominator of b''. By Claim 6.2. there exist $2hr$ internally disjoint (y, b'')-paths of length $h-i+1$. Since $2hr > |V(P^+ \cup R \cup R_0)|$, one of these paths, say R', is internally disjoint from $P^+ \cup R \cup R_0$. Now $P^+ \cup R \cup R_0 \cup R'$ is a linear cycle of length $2m - 2h + 2(i-j) + h - i + 2j + h - i + 1 = 2m+1$ in G, a contradiction. This completes Case 2.

Case 3. $|B_i'| \geqslant B^+/2h$ for some $1 \leqslant i \leqslant h-1$.

The arguments for this case are similar to those for Case 2, except that the argument for condition (iii) is more delicate. As in Case 2, we define Γ and Γ_x analogously, with L_i being replaced by L_i' in the definitions. Let $L_{i,1}' = \{x \in L_i', x \text{ is heavy and is of type } 1\}$. Let $L_{i,2}' = \{x \in L_i', x \text{ is heavy and is of type } 2\}$. We only need to modify the argument for the statement that whenever $x, y \in L_{i,1}', x \neq y$ and $e \in \Gamma_x$ and $f \in \Gamma_y$ we have $\phi(x) \cap \phi(y) = \emptyset$.

Suppose for contradiction that $\phi(e) \cap \phi(f) \neq \emptyset$. Let $v \in \phi(e) \cap \phi(f)$. Then $\phi(e) \cap \phi(f) = \{v\}$. Suppose $e = ab$ and $f = a'b'$, where $a \in A_x, b \in C_x$ and $a' \in A_y, b' \in C_y$. Let $\ell = 2m - 2 - 2h$. Since $\delta(\Gamma_y) \geqslant 2mr > r\ell + 2r$, by Lemma 3.4, there exists a path P in Γ_y of length ℓ starting at b' that is strongly rainbow under ϕ and such that

$$\left(\bigcup_{e' \in P} \phi(e')\right) \cap (\phi(e) \cup \phi(f)) = \emptyset.$$

Let b'' denote the other endpoint of P. Since P has even length, $b'' \in C_y$. Let P^+ denote the set of the edges of E that correspond to the edges of $P \cup \{e, f\}$. Then P^+ is linear path of length $2m - 2h$ in G where a is an endpoint at one end and b'' is an endpoint at the other end. Furthermore, $V(P^+) \cap V(H) \subseteq L_h$. Let R be a monotone path in H from x to a. Then R has length $h-i$ and is internally disjoint from P^+. Since $b'' \in C_y$, y is a dominator of b''. By Claim 6.2, there exist $2hr$ internally disjoint (y, b'')-paths of length $h-i+1$ contained in $(\bigcup_{j=i}^{h-1} H_j) \cup E$. Since $2hr > |V(P^+ \cup R)|$, one of these paths, say R', is internally disjoint from $P^+ \cup R$. Now $W = P^+ \cup R \cup R'$ is a linear (x, y)-path of length $2m - 2i + 1$ contained in $(\bigcup_{j=i}^{h-1} H_j) \cup E$. Let e_x denote the edge of W containing x and let e_y denote the edge of W containing y. Each of e_x, e_y intersects L_i at exactly one vertex. Suppose $e_x \cap L_i = \{x^*\}$ and $e_y \cap L_i = \{y^*\}$. Then

$$V(W) \cap \left(\bigcup_{j=0}^{i} V(H_j)\right) = \{x^*, y^*\}.$$

By Lemma 5.2 there is an (x^*,y^*)-path R_0 of length $2t \leqslant 2i$ in $\bigcup_{j=0}^{i} H_j$ such that $V(R_0)\cap L_i = \{x^*,y^*\}$. If $t=i$, then $W\cup R_0$ is a linear cycle in G of length $2m+1$, a contradiction. So suppose $t<i$. The idea now is to keep R,R_0 and e_y and redefine P and R' to get a linear cycle of length $2m+1$. Let $\ell = 2m-2h+2(i-t)-3$. Note that $\ell>0$ and is odd. Since $\delta(\Gamma_y)\geqslant 2mr > r\ell+2r$, by Lemma 3.4, there exists a path P in Γ_y of length ℓ starting at a' that is strongly rainbow under ϕ and such that

$$\left(\bigcup_{e'\in P}\phi(e')\right)\cap(\phi(e)\cup\phi(f))=\emptyset.$$

Let b'' denote the other endpoint of P. Since ℓ is odd, $b''\in C_y$. Let P^+ denote the set of the edges of E that correspond to the edges of $P\cup\{e,f\}$. Then P^+ is linear path of length $2m-2h+2(i-t)-1$ in G where a is an endpoint at one end and b'' is an endpoint at the other end. Furthermore, $V(P^+)\cap V(H)\subseteq L_h$. As before, there are $2hr$ internally disjoint (y,b'')-paths of length $h-i+1$ contained in $(\bigcup_{j=i}^{h-1} H_j)\cup E$. Since $2hr > |V(P^+\cup R\cup e_y)|$, one of these paths, say R', is internally disjoint from $P^+\cup R\cup e_y$. It is also internally disjoint from R_0 by the definition of R_0. Now $P^+\cup R\cup R_0\cup\{e_y\}\cup R'$ is a linear cycle of length $2m+1$ in G, a contradiction. This completes Case 3 and the proof of the lemma. □

Theorem 6.3. *Let m,r be positive integers where $m\geqslant 2$ and $r\geqslant 3$. There exist a positive real $c'=c'_{m,r}$ and a positive integer n_2 such that for all $n\geqslant n_2$ we have $\mathrm{ex}_L(n,C^r_{2m+1})\leqslant c'n^{1+1/m}$.*

Proof. We follow the steps in Theorem 4.4, using Lemma 6.1 in place of Lemma 4.1. Let $\lambda = 2m^2r^2$. Let $c=2^{r+2}\lambda^m$ as in Lemma 6.1. Let $c'_{m,r}=2m^{r-1}c^m$. Choose n_2 such that $c'_{m,r}n_2^{1/m}\geqslant n_0$, where n_0 is given in Lemma 3.8. Let G be an n-vertex linear r-graph with at least $c'_{m,r}n^{1+1/m}$ edges, where $n\geqslant n_2$. Suppose that G does not contain a copy of C^r_{2m+1}, we derive a contradiction. By our assumption, G has average degree at least $rc'_{m,r}n^{1/m}$. By Lemma 3.6, there exists a subgraph G_0 of G with $\delta(G_0)\geqslant c'_{m,r}n^{1/m}$. Let $N=n(G_0)$. Then $N\geqslant c'_{m,r}n^{1/m}\geqslant n_0$ and $\delta(G')\geqslant c'_{m,r}N^{1/m}$. By Lemma 3.8 (with $t=m$), there exists a partition of $V(G')$ into $S_0,\dots,S_{m-1}$ such that for each $u\in V(G')$ and $i\in\{0,\dots,m-1\}$, we have

$$|L_{G'}(u)\cap G'[S_i]|\geqslant \frac{c'_{m,r}}{2m^{r-1}}N^{1/m}=c^mN^{1/m}.$$

Let w be any vertex in S_0. Let $L_0=\{w\}$. Inside G', we will construct a levelled linear quasi-tree H of height m rooted at w with segments $H_0,\dots,H_{m-1}$ and main levels $L_0,L_1,\dots,L_m$ such that, for all $i\in\{0,\dots,m-1\}$, $V(H_i)\subseteq S_i$. (Note that this means that for all $i\in[m]$, $L_i\subseteq S_{i-1}$.) Furthermore, we will maintain that for all $i\in[m]$, $|L_i|\geqslant N^{1/m}|L_{i-1}|$. This will imply that $|L_m|\geqslant N$, a contradiction.

We construct H as follows. Let H_0 consist of the edges of $G'[S_0]$ containing w. By our assumption, $|H_0|\geqslant c^mN^{1/m}\geqslant N^{1/m}$, by our definition of c. Let L_1 consist of a vertex from $e\setminus\{w\}$ for each $e\in H_0$. We have $|L_1|=|H_0|\geqslant N^{1/m}|L_0|$. In general, suppose $1\leqslant i\leqslant m-1$ and suppose we have defined $H_0,\dots,H_{i-1}$ and $L_0,L_1,\dots,L_i$ that satisfy the requirements. Let E denote the set of edges in G' that contain one vertex in $L_i\subseteq S_{i-1}$ and $r-1$ vertices in S_i. By the definition of the partition $(S_0,\dots,S_{m-1})$, $|E|\geqslant c^mN^{1/m}|L_i|\geqslant c^i|L_i|$. Since $C^r_{2m+1}\not\subseteq G'$, by Lemma 6.1, there exists a subset $E^*\subseteq E$ such that (i) $|E^*|\geqslant(1/c^i)|E|$, and (ii) E^* is the r-expansion of some bipartite

2-graph Γ with one part in L_i and the other part outside $\bigcup_{j=0}^{i-1} H_{i-1}$. Now, let H_i be the r-graph formed by E^* and let L_i consist of one vertex from $e \setminus V(\Gamma)$ for each $e \in E^*$ (note that this implies that $|L_i| = |E^*|$). Now, $\bigcup_{j=0}^{i} H_i$ is a levelled linear quasi-tree in G' rooted at w with height i and main levels $L_0, L_1, \ldots, L_i$. Furthermore,

$$|L_i| = |E^*| \geqslant \frac{1}{c^i}|E| \geqslant \frac{c^m}{c^i} N^{1/m}|L_i| \geqslant N^{1/m}|L_i|.$$

We can continue for m steps to obtain $|L_m| \geqslant N$, which yields the desired contradiction. $\square$

7. Cycle-complete Ramsey numbers

Given two r-graphs G and H, the *Ramsey number* $R(G,H)$ is the smallest positive integer n such that in every colouring of the edges of K_n^r using two colours red and blue, there exists either a red copy of G or a blue copy of H. As mentioned in the Introduction, part of the motivation behind our study of the linear Turán number of linear cycles comes from the study by Kostochka, Mubayi and Verstraëte [29] on the hypergraph Ramsey number of a linear triangle versus a complete graph. Their work is in part inspired by the study of graph Ramsey number $R(C_3, K_t)$. A celebrated result of Kim [28] together with earlier upper bounds by Ajtai, Komlós and Szemerédi [1] shows that

$$R(C_3, K_t) = \Theta\left(\frac{t^2}{\log t}\right), \quad \text{as } t \to \infty.$$

Kostochka, Mubayi and Verstraëte obtained the following bounds.

Theorem 7.1 ([29]). *There exist constants $a, b_r > 0$ such that, for all $t \geqslant 3$,*

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leqslant R(C_3^3, K_t^3) \leqslant b_3 t^{3/2},$$

and, for $r \geqslant 4$,

$$\frac{t^{3/2}}{(\log t)^{3/4+o(1)}} \leqslant R(C_3^r, K_t^r) \leqslant b_r t^{3/2}.$$

Theorem 7.2 ([29]). *For fixed $r, k \geqslant 3$,*

$$R(C_k, K_t^r) = \Omega^*(t^{1+1/(3k-1)}), \quad \text{as } t \to \infty.$$

There exists a constant $c_r > 0$ such that

$$R(C_5, K_t^r) \geqslant c_r \left(\frac{t}{\ln t}\right)^{5/4}, \quad \text{as } t \to \infty.$$

Here the authors use $f = O^*(g)$ to denote that for some constant $c > 0$, $f(t) = O((\ln t)^c g(t))$ and $f = \Omega^*(g)$ is equivalent to $g = O^*(f)$. The key point of Theorem 7.2 is that the exponent $1 + 1/(3k-1)$ of t is bounded away from 1 by a constant independent of r. The authors made the following conjecture.

Conjecture 7.3 ([29]). *For all fixed $r \geqslant 3$, $R(C_3, K_t^r) = o(t^{3/2})$ and $R(C_5, K_t^r) = O(t^{5/4})$, as $t \to \infty$.*

Using our bounds on the linear Turán numbers, we can quickly derive non-trivial upper bounds on $R(C_\ell^r, K_t^r)$ for all $r, \ell \geqslant 3$. First, however, let us recall some results on cycle-complete Ramsey numbers of graphs. As mentioned above, the behaviour of $R(C_3, K_t)$ is now quite well understood, particularly with the recent deep work of [6] and [17]. For the general cycle-complete Ramsey numbers, the best known upper bound on even cycles versus cliques is

$$R(C_{2m}, K_t) = O\left(\left(\frac{t}{\ln t}\right)^{m/(m-1)}\right),$$

due to Caro, Li, Rousseau and Zhang [11]. The best known upper bound on odd cycles versus cliques is

$$R(C_{2m+1}, K_t) = O\left(\frac{t^{(m+1)/m}}{(\ln t)^{1/m}}\right),$$

due to Sudakov [39] and Li and Zang [26]. The best known lower bound is

$$R(C_\ell, K_t) = \Omega\left(\frac{t^{(\ell-1)/(\ell-2)}}{\ln t}\right),$$

due to Bohman and Keevash [5].

We now obtain some upper bounds on $R(C_\ell^r, K_t^r)$ using linear Turán numbers and a reduction process via the well-known Sunflower Lemma. A sunflower (or Δ-system) $\mathcal{F}$ with core C is a collection of distinct sets $A_1, \ldots, A_p$ such that for all $i, j \in [p]$ we have $A_i \cap A_j = C$. We call the A_i *members* of the sunflower. If a sunflower has p members and the core has size a, then we call it an (a, p)-sunflower. Note that the core is allowed to be empty and hence a matching is considered to be a sunflower.

Lemma 7.4 (Sunflower Lemma [14]). *If $\mathcal{F}$ is a collection of sets of size at most k and $|\mathcal{F}| \geqslant k!(p-1)^k$, then $\mathcal{F}$ contains a sunflower with p members.*

Partly following the approach in [29], we consider non-uniform hypergraphs, but will disallow singletons as edges. Recall that a *linear cycle* of length ℓ is a list of sets $A_1, \ldots, A_\ell$ such that $|A_i \cap A_{i+1}| = 1$ for $i = 1, \ldots, \ell - 1$, $|A_\ell \cap A_1| = 1$ and $A_i \cap A_j = \emptyset$ for all other pairs i, j, $i \neq j$. A set S in a hypergraph G is an independent set in G if no edge of G is contained in S. Let $\alpha(G)$ denote the maximum size of an independent set in G. The next lemma is in spirit similar to a sequence of lemmas given in Section 3.1 of [29], except that here we use the Sunflower Lemma. A hypergraph is *simple* if no edge contains another edge.

Lemma 7.5. *Let $m, r \geqslant 2$ be integers. Let G be a hypergraph whose edges have sizes between 2 and r. Suppose G does not contain a linear cycle of length ℓ. Then there exists a simple hypergraph G' on $V(G)$ whose edges have sizes between 2 and r such that G' contains no linear cycle of length ℓ, G' contains no $(a, r\ell)$-sunflower for any $a \geqslant 2$, and $\alpha(G') \leqslant \alpha(G)$.*

Proof. We iterate the following process. Let $\mathcal{F}$ be an $(a, r\ell)$-sunflower in G with core C, where $|C| = a \geqslant 2$. Let G_1 be obtained from G by replacing some edge e in $\mathcal{F}$ with C. If G_1 contains a linear cycle L of length ℓ, then L must use C as an edge. Since L contains at most $r\ell$ vertices and C is the core of a sunflower $\mathcal{F}$ with $r\ell$ members, we can find some edge e' in $\mathcal{F}$ such that $e' \setminus C$ is disjoint from $V(L)$. Now if we replace C with e' in L, we obtain a linear cycle of length ℓ in G, a contradiction. So, G_1 has no linear cycle of length ℓ. Clearly, any independent set S in G is also an independent set in G_1. So $\alpha(G_1) \leqslant \alpha(G)$. We now replace G with G_1 and repeat this process until there is no longer an $(a, r\ell)$-sunflower for some $a \geqslant 2$. The process must end since the total edge-size decreases at each step. Denote the final graph by G'. If G' is not simple then we make it simple by removing edges that contain other edges. This cannot create a linear cycle of length ℓ, or a new sunflower, or increase the independence number. Then G' satisfies the claim. □

A hypergraph G is $(2,q)$-*linear* if no pair of vertices is contained in q or more edges of G.

Lemma 7.6. *Let $a, p, r \geqslant 2$ be integers. Let G be a simple hypergraph whose edges have sizes between 2 and r and that contains no (a, p)-sunflower for any $a \geqslant 2$. Then G is $(2,q)$-linear, where $q = r!(p-1)^r$.*

Proof. Otherwise some pair $\{a, b\}$ would be contained in a set H of at least q edges of G. Let $H' = \{e \setminus \{a, b\} : e \in H\}$. Since $H \subseteq G$ is simple, $|H'| = |H| \geqslant q = r!(p-1)^r$. By Lemma 7.4, H' contains a sunflower $\mathcal{F}$ with p members. Now, adding $\{a, b\}$ to each member of $\mathcal{F}$ yields an (a, p)-sunflower in G, where $a \geqslant 2$, contradicting our assumption about G. □

Lemma 7.7. *Let $r \geqslant 2, q$ be positive integers. Let G be a hypergraph whose edges have sizes between 2 and r. Suppose G is $(2,q)$-linear. Then G contains a linear subgraph G' with $|G'| \geqslant (2/qr^2)|G|$.*

Proof. By our assumption, each edge e of G shares a pair of vertices with at most $\binom{r}{2}(q-1)$ other edges. Let H be a graph whose vertices are the edges of G such that two vertices u, v are adjacent in H if the corresponding edges in G share a pair of vertices. Then $\Delta(H) < \binom{r}{2}q - 1$. Hence H contains an independent set S of size at least

$$\frac{|V(H)|}{\Delta(H)+1} \geqslant \frac{2|V(H)|}{qr^2}.$$

Let G' be the subgraph of G whose edges correspond to S. Then G' is a linear subgraph of G with $|G'| \geqslant (2/qr^2)|G|$. □

Lemma 7.8. *Let H be a linear hypergraph whose edges have sizes between 2 and r. Suppose H does not contain a linear cycle of length ℓ. Let $D = \partial_2(H)$. Let v be any vertex in $V(D) = V(H)$. Then $|D[N_H(v)]| \leqslant r^{r+4}\ell|N_H(v)|$.*

Proof. Since H is linear, the link graph $\mathcal{L}_H(x)$ consists of disjoint edges each of size at most $r-1$. Let $U = V(\mathcal{L}_H(v)) = N_H(v)$. The edges of $\mathcal{L}_H(x)$ form a partition of U into parts of size at most $r-1$ (with each part being an edge of $\mathcal{L}_H(x)$). Also, since H is linear, no edge of H

contains more than one vertex from any of those parts. Let us randomly and independently pick one vertex from each part, and call the resulting set S. For each edge in $H[U]$ the probability of it being in $H[S]$ is at least $(1/(r-1))^r$. So there is a choice of S for which

$$H[S] \geqslant \frac{1}{(r-1)^r}|H[U]|.$$

If $H[S]$ has average degree at least $r^2\ell$, then it contains a subgraph H' with minimum degree at least $r\ell$, and since H' is linear, one can easily find a linear path P of length $\ell-2$, say, with endpoint a and b. Let e_a be the edge of H that contains $\{x,a\}$ and let e_b be the edge of H that contains $\{x,b\}$. Then $e_a \cap S = \{a\}, e_b \cap S = \{b\}$. In particular, we see that $P \cup \{e_a, e_b\}$ is a linear cycle of length ℓ, a contradiction. So $H[S]$ has average degree less than $r^2\ell$. Therefore

$$|H[U]| \leqslant (r-1)^r|H[S]| < r^r\frac{r^2}{2}\ell|S| < r^{r+2}\ell|U|,$$

and hence

$$|D[U]| \leqslant \binom{r}{2}|H[U]| < r^{r+4}\ell|U|. \qquad \square$$

We need the following lemma due to Alon [2]. The version stated below is implicit in the proof of Proposition 2.1 in [2]. Alternatively, one could also apply Theorem 1.1 of [3]. Logarithms below are in base 2.

Lemma 7.9 ([2]). *Let G be a graph with maximum degree at most $d \geqslant 1$, in which for any vertex v, $G[N(v)]$ contains an independent set of size at least $|N(v)|/p$. Then*

$$\alpha(G) \geqslant \frac{n\log d}{160d\log(p+1)}.$$

Theorem 7.10. *Let m,r be integers where $m \geqslant 2$ and $r \geqslant 3$. There exists a constant $a_{m,r}$, depending on m and r, such that*

$$R(C^r_{2m}, K^r_t) \leqslant a_{m,r}\left(\frac{t}{\ln t}\right)^{m/(m-1)}.$$

Proof. The definition of $a_{m,r}$ depends on various constants we defined earlier and will be implicit in our proof. Let

$$n \geqslant a_{m,r}\left(\frac{t}{\ln t}\right)^{m/(m-1)}.$$

By choosing $a_{m,r}$ to be large enough, we may assume that $n \geqslant n_1$, where n_1 is given in Theorem 4.4. It suffices to show that if G is an n-vertex r-graph that does not contain C^r_{2m} then G contains an independent set of size at least t. Let such G be given. By Lemma 7.5, there exists a simple hypergraph G' with $V(G') = V(G)$ such that $\alpha(G') \leqslant \alpha(G)$, G' contains no linear cycle of length $2m$, and that G' contains no $(a, 2mr)$-sunflower for any $a \geqslant 2$. By Lemma 7.6, G' is $(2,q)$-linear, where $q = r!(2mr-1)^r$. By Lemma 7.7, G' contains a linear subgraph with $|G''| \geqslant c_1|G'|$, where c_1 is a positive constant depending on m and r. Clearly, G'' contains no linear cycle of length $2m$. Applying the $O(n^{1+1/m})$ bound [7] on $\mathrm{ex}(n,C_{2m})$ and Theorem 4.4, by considering edges of various sizes, we have $|G''| \leqslant c_2 n^{1+1/m}$, for some constants c_2, depending on m and r. Hence $|G'| \leqslant c_3 n^{1+1/m}$ for some constant c_3, depending on m and r. So G' has average degree at most $rc_3 n^{1/m}$. Clearly, at most $n/2$ vertices in G' can have degree at least $2rc_3 n^{1/m}$. Let H be the subgraph of G' induced by vertices of degree at most $2rc_3 n^{1/m}$. Then $|V(H)| \geqslant n/2$ and $\Delta(H) \leqslant 2rc_3 n^{1/m}$.

Let $D = \partial_2(H)$. Then $\Delta(D) \leqslant 2r^2 c_3 n^{1/m}$. Note that for each vertex v we have $N_D(v) = N_H(v)$, which we will denote by $N(v)$. As H does not contain a linear cycle of length $2m$, by Lemma 7.8, for each vertex v in D, we have $|D[N(v)]| \leqslant 2mr^{r+4}|N(v)|$. So $D[N(v)]$ has average degree at most $4mr^{r+4}$. By a well-known result of Caro and Wei [10, 41], $D[N(v)]$ contains an independent set of size at least

$$\frac{|N(v)|}{4mr^{r+4}+1}.$$

By Lemma 7.9, with $d = 2r^2 c_3 n^{1/m}$,

$$\alpha(D) \geqslant c_5 \frac{n \ln n}{n^{1/m}} = c_5 n^{(m-1)/m} \ln n,$$

for some positive constant c_5, depending on m and r. Since

$$n \geqslant a_{m,r} \left(\frac{t}{\ln t}\right)^{m/(m-1)},$$

by choosing $a_{m,r}$ to be large enough, we can ensure $\alpha(D) \geqslant t$. Certainly any independent set in D is also an independent set in G'. Hence $\alpha(G') \geqslant t$ and $\alpha(G) \geqslant \alpha(G') = t$. $\square$

For odd cycle-complete Ramsey numbers, we need some more definitions and a lemma. Let H be a hypergraph whose vertices are ordered by a total order π. Let P be a linear path of length ℓ, that is, P consists of a list of edges $e_1, \ldots, e_\ell$ such that $|e_i \cap e_{i+1}| = 1$ for each $i \in [\ell-1]$ and $e_i \cap e_j = \emptyset$ whenever $|i-j| > 1$. For each $i \in [\ell-1]$, let $e_i \cap e_{i+1} = \{x_i\}$. We say that P is an *increasing linear path* under π if $\pi(v) < \pi(x_1)$ for all $v \in e_1 \setminus \{x_1\}$, $\pi(x_{\ell-1}) < \pi(v)$ for all $v \in e_\ell \setminus x_{\ell-1}$, and we have $\pi(x_{i-1}) < \pi(v) < \pi(x_i)$ for each $i = 2, \ldots, \ell-1$ and $v \in e_i \setminus \{x_{i-1}, x_i\}$. If P is an increasing linear path and v is the largest vertex on P under π, then we say that P *ends* at v.

Lemma 7.11. *Let H be a hypergraph and π a total order on $V(H)$. If H does not contain an increasing linear path of length ℓ, then $V(H)$ can be partitioned into ℓ independent sets.*

Proof. For each $i = 0, \ldots \ell-1$, let S_i denote the set of vertices v such that the longest increasing linear path in H that ends at v has length i. Then $S_0, \ldots, S_{\ell-1}$ partition $V(H)$. Suppose for some $i \in \{0, \ldots, \ell-1\}$, S_i contains an edge e. Let v and v' be the vertices in e that are smallest and largest under π, respectively. By definition, H contains an increasing linear path P of length i that ends at v. Now $P \cup e$ is an increasing path of length $i+1$ that ends at v', contradicting $v' \in S_i$. Hence for each i, S_i contains no edge of H and hence is an independent set in H. $\square$

The following lemma is a variant of Theorem 1 in [13]. The proof is similar.

Lemma 7.12. *Let H be a hypergraph whose edges have sizes between 2 and r. Suppose H does not contain a linear cycle of length $2m+1$. Let H^* be the subgraph of H consisting of all the edges of size 2 in H. Let $v \in V(G)$. For each i, let S_i be the set of vertices in H^* that are at distance i from v. Then for each $i \leqslant m$, $H[S_i]$ contains an independent set of size at least $|S_i|/(2m-1)$.*

Proof. Grow a breadth-first search tree T in H^* from v. So the levels of T are precisely the distance classes from v in H^*. For each $i \geqslant 1$, define a linear order π_i of S_i as follows. Let π_1 be an arbitrary linear order on S_1. For each $i \geqslant 2$, let π_i be a linear order on S_i obtained by listing the children of the first vertex in π_{i-1}, followed by the children of the second vertex in π_{i-1}, *etc.* For each $1 \leqslant i \leqslant m$, we claim that $H[S_i]$ contains no increasing linear path of length $2m-1$. Otherwise, fix an i for which $H[S_i]$ contains an increasing linear path P of length $2m-1$ with edges $e_1, e_2, \ldots, e_{2m-1}$ in order. Let x_1 be the least vertex in e_1 under π_i. Let x_{2m} be the largest vertex in e_{2m-1} under π_i. For each $k \in \{2, \ldots, 2m-1\}$, let $e_{k-1} \cap e_k = \{x_k\}$. Then $x_1 < x_2 < \cdots < x_{2m}$ in π_i. Let w be a closest common ancestor of $x_1, \ldots, x_{2m}$ in T. Suppose $w \in S_j$, where $j < i$. Let k be the smallest positive integer such that x_k and x_{k+1} are under different children of w. Such a k exists by our choice of w. By our ordering on each level, the ancestors of $x_1, \ldots, x_k$ in S_j precede ancestors of $x_{k+1}, \ldots, x_{2m}$ in S_j under π_j. Hence for any $a \in [k]$, $b \in [2m] \setminus [k]$, the unique (x_a, x_b)-path $Q_{a,b}$ in T must pass through w and has length $2(i-j)$. Based on the value of k, we can find $a \in [k]$, $b \in [2m] \setminus [k]$ such that $b - a = 2m+1-2(i-j)$. Now $Q_{a,b} \cup \{e_a, e_{a+1}, \ldots, e_{b-1}, e_b\}$ is a linear cycle of length $2m+1$ in H, a contradiction. Hence $H[S_i]$ contains no increasing linear path of length $2m-1$. By Lemma 7.11, $H[S_i]$ contains an independent set of size at least $|S_i|/(2m-1)$. □

Theorem 7.13. *Let m, r be positive integers where $m \geqslant 2, r \geqslant 3$. There exists a positive constant $b_{m,r}$, depending on r and m, such that $R(C^r_{2m+1}, K^r_t) \leqslant b_{m,r} t^{m/(m-1)}$.*

Proof. Our choice of $b_{m,r}$ will depend on other constants defined earlier and will be implicit in the proof. Let $n \geqslant b_{m,r} t^{m/(m-1)}$. By choosing $b_{m,r}$ to be large enough, we may assume that $n \geqslant n_2$, where n_2 is specified in Theorem 6.3. Let G be any n-vertex r-graph on n vertices not containing a copy of C^r_{2m+1}. We show that G contains an independent set of size at least t.

By Lemma 7.5, there exists a simple hypergraph G' on $V(G)$ whose edges have sizes between 2 and r such that $\alpha(G') \leqslant \alpha(G)$, G' contains no linear cycle of length $2m+1$, and that G' contains no $(a, (2m+1)r)$-sunflower for any $a \geqslant 2$. By Lemma 7.6, G' is $(2,q)$-linear where $q = r![(2m+1)r-1]^r$. For each $j = 3, \ldots, r$, let G_j denote the subgraph of G' consisting of edges of size j. Let $G'' = \bigcup_{j=3}^r G_j$. Then G'' is $(2,q)$-linear. By Lemma 7.7, G'' contains a linear subgraph G^* with $|G^*| \geqslant (2/qr^2)|G''|$. By Theorem 6.3, $|G^*| \leqslant c'_1 n^{1+1/m}$ for some positive constant c'_1 depending on m and r. Hence $|G''| \leqslant c'_2 n^{1+1/m}$ for some positive constant c'_2 depending on m and r. The number of vertices of G'' of degree at least $2rc'_2 n^{1/m}$ is at most $n/2$. Let U be the set of vertices of degree at most $2rc'_2 n^{1/m}$ in G''. Then $|U| \geqslant n/2$. Let $H = G'[U]$. Let H^* be the subgraph of H consisting of edges of size 2. Let H' be subgraph of H consisting of edges of size 3 or more. By our definition of H, $\Delta(H') \leqslant 2rc'_2 n^{1/m}$. We obtain a large independent set W in H as follows. Initially set $W = \emptyset$. Let v be any vertex in H, and for each $i \geqslant 2$ let S_i denote the set of vertices at distance i from v in H^*. Let $k \in [m-1]$ be the smallest integer such that $|S_{i+1}|/|S_i| \leqslant n^{1/m}$. Such a k exists since otherwise we would have $|S_m| > n$, a contradiction. Since H contains no linear cycle of length $2m+1$, by Lemma 7.12, $H[S_k]$ contains an independent set S' of size at least $|S_k|/(2m-1)$. Let $\widetilde{S} = S_{k-1} \cup S_k \cup S_{k+1}$. Then the neighbours in H^* of vertices in S' lie in $\widetilde{S}$. By our choice of k,

$$|\widetilde{S}| < (n^{1/m}+2)|S_k| < (2m-1)(n^{1/m}+2)|S'| < 3mn^{1/m}|S'|.$$

Let Z be a set of vertices in H obtained by picking a vertex in $e \setminus \widetilde{S}$, if it exists, for each edge e in H' that contains a vertex in S'. Since $\Delta(H') \leqslant 2rc_2'n^{1/m}$, we have $|Z| \leqslant 8rc_2'n^{1/m}|S'|$. By our discussion above, $|\widetilde{S} \cup Z| \leqslant c_3'n^{1/m}|S'|$ for some positive constant c_3' depending on m and r. We add S' to U and delete $\widetilde{S} \cup Z$ from H, and iterate the process until we run out of vertices. By design, the final W is an independent set in H that has size at least

$$\frac{n/2}{c_3'n^{1/m}} \geqslant \frac{n^{(m-1)/m}}{2c_3'}.$$

Since $n \geqslant b_{m,r}t^{m/(m-1)}$, by choosing $b_{m,r}$ to be large enough, we can ensure $\alpha(H) \geqslant t$. Since $H = G'[U]$, we have $\alpha(G) \geqslant \alpha(G') \geqslant t$. □

8. Concluding remarks

Our main objective in this paper has been to establish an upper bound on $\mathrm{ex}_L(n, C_\ell^r)$ of the form $O(n^{1+1/\lfloor \ell/2 \rfloor})$. We chose constants $c_{r,\ell}$ and $c'_{r,\ell}$ in Theorem 4.4 and Theorem 6.3 larger than necessary in order to simplify our presentation. Motivated by the known bounds on graph even cycles, we would like to ask the following question.

Problem 8.1. *Determine whether or not there exists a constant $c(r)$, depending only on r, such that* $\mathrm{ex}_L(n, C_{2m}^r) \leqslant c(r)mn^{1+1/m}$.

The study of $\mathrm{ex}_L(n, C_{2m}^3)$ has a natural connection to the so-called *rainbow Turán number* $\mathrm{ex}^*(n, C_{2m})$ of a cycle of length $2m$, which denotes the maximum number of edges in an n-vertex graph that admits a proper edge-colouring that contains no cycle of length $2m$ all of whose edges have different colours. The main conjecture from [27] is that $\mathrm{ex}^*(n, C_{2m}) = O(n^{1+1/m})$, which remains open except for C_4 and C_6. See Das, Lee and Sudakov [12] for some recent progress on the problem. Interestingly, there it is not too hard to obtain an $\Omega(n^{1+1/m})$ lower bound on $\mathrm{ex}^*(n, C_{2m})$ through an explicit construction using B_k^*-sets. Here, we are able to establish an $O(n^{1+1/m})$ upper bound on $\mathrm{ex}_L(n, C_{2m}^r)$ and $\mathrm{ex}_L(n, C_{2m+1}^r)$, but no good lower bounds on $\mathrm{ex}_L(n, C_\ell^r)$ are currently known, except for C_3^3, C_4^3 and C_5^3. Using generalized Sidon sets such as those considered in [37] and [32], one could obtain some non-trivial lower bounds on $\mathrm{ex}_L(n, C_\ell^r)$. This is an area worth exploring.

Our Ramsey bounds on $R(C_\ell^r, K_t^r)$ are similar to those for graphs. However, as speculated in [29], for $r \geqslant 3$ perhaps $R(C_\ell^r, K_t^r) = \Theta^*(t^{\ell/(\ell-1)})$ holds, where O^* and Ω^* are defined in Section 7. It will be interesting to further sharpen our bounds on $R(C_\ell^r, K_t^r)$. By analysing the proof of Theorem 6.3, together with Lemma 7.8 and Lemma 7.9, one might be able to improve our bound on $R(C_{2m+1}^r, K_t^r)$ by a factor of $(\ln t)^c$. However, since the exponent $m/(m-1)$ is likely not the correct one, we have made no such attempt. Since the submission of our original paper, A. Méroueh [33] has made some improvements upon our bounds. Using our linear Turán bounds, it is not hard to show that $RL(C_\ell^r, K_t^r) = O^*(t^{\ell/(\ell-1)})$, where $RL(C_\ell^r, K_t^r)$ is defined to be the smallest n such that every linear r-graph not containing C_ℓ^r has an independent set of size t (see Theorem 1.2 of [29] for a proof for C_3^3).

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