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New bounds for a hypergraph bipartite Turán problem

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ABSTRACT

Let t be an integer such that $t \geq 2$. Let $K_{2,t}^{(3)}$ denote the triple system consisting of the $2t$ triples $\{a, x_i, y_i\}$, $\{b, x_i, y_i\}$ for $1 \leq i \leq t$, where the elements $a, b, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$ are all distinct. Let $\mathrm{ex}(n, K_{2,t}^{(3)})$ denote the maximum size of a triple system on n elements that does not contain $K_{2,t}^{(3)}$. This function was studied by Mubayi and Verstraëte [9], where the special case $t = 2$ was a problem of Erdős [1] that was studied by various authors [3,9,10].

Mubayi and Verstraëte proved that $\mathrm{ex}(n, K_{2,t}^{(3)}) < t^4\binom{n}{2}$ and that for infinitely many n, $\mathrm{ex}(n, K_{2,t}^{(3)}) \geq \frac{2t-1}{3}\binom{n}{2}$. These bounds together with a standard argument show that $g(t) := \lim_{n\to\infty} \mathrm{ex}(n, K_{2,t}^{(3)})/\binom{n}{2}$ exists and that

$$\frac{2t-1}{3} \leq g(t) \leq t^4.$$

Addressing the question of Mubayi and Verstraëte on the growth rate of $g(t)$, we prove that as $t \to \infty$,

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$g(t) = \Theta(t^{1+o(1)})$.

1. Introduction

An r-*graph* is an r-uniform hypergraph. Let $\mathcal{F}$ be a family of r-graphs and let $\mathrm{ex}(n, \mathcal{F})$ denote the maximum number of edges in an r-graph on n vertices containing no member of $\mathcal{F}$. We call $\mathrm{ex}(n, \mathcal{F})$ the *Turán number* of $\mathcal{F}$. When $\mathcal{F}$ consists of a single graph F, we write $\mathrm{ex}(n, F)$ for $\mathrm{ex}(n, \mathcal{F})$. When $r \geq 3$, determining $\mathrm{ex}(n, \mathcal{F})$ asymptotically or exactly is notoriously difficult. Katona, Nemetz, and Simonovits [7] showed that $\lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ exists and this limit is called the *Turán density* of $\mathcal{F}$, and is denoted by $\pi(\mathcal{F})$. When $\pi(\mathcal{F}) = 0$, that is, when $\mathrm{ex}(n, \mathcal{F}) = o(n^r)$, we call the problem of determining $\mathrm{ex}(n, \mathcal{F})$ a *degenerate hypergraph Turán problem*. For an excellent survey on the study of hypergraph Turán numbers, see [8]. In this paper, we study a degenerate hypergraph Turán problem that is motivated by the study of Turán numbers of complete bipartite graphs as well as by a question of Erdős. In fact, the r-graph F we study in this paper satisfies $\mathrm{ex}(n, F) = \Theta(n^{r-1})$, so in this case, the natural goal is to determine $\lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{r-1}$.

Definition 1. Let $r \geq 3$ be an integer. Let G be a bipartite graph with an ordered bipartition (X, Y). Suppose that $Y = \{y_1, \dots, y_m\}$. Let $Y_1, \dots, Y_m$ be disjoint sets of size $r - 2$ that are disjoint from $X \cup Y$. Let $G^{(r)}_{X,Y}$ denote the r-graph with vertex set $(X \cup Y) \cup (\bigcup_{i=1}^m Y_i)$ and edge set $\bigcup_{i=1}^m \{e \cup Y_i : e \in E(G), y_i \in e\}$.

Let $s, t \geq 2$ be positive integers. If G is the complete bipartite graph with an ordered bipartition (X, Y) where $|X| = s, |Y| = t$, then let $G^{(r)}_{X,Y}$ be denoted by $K^{(r)}_{s,t}$.

Definition 2. For all $n \geq r \geq 3$, let $f_r(n)$ denote the maximum number of edges in an n-vertex r-graph containing no four edges A, B, C, D with $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$.

Note that $f_3(n) = \mathrm{ex}(n, K^{(3)}_{2,2})$, and in general $f_r(n) \leq \mathrm{ex}(n, K^{(r)}_{2,2})$. Erdős [1] asked whether $f_r(n) = O(n^{r-1})$ when $r \geq 3$. Füredi [3] answered Erdős' question affirmatively. More precisely, he showed that for integers n, r with $r \geq 3$ and $n \geq 2r$,

$$\binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor \leq f_r(n) < 3.5\binom{n}{r-1}. \tag{1}$$

The lower bound is obtained by taking the family of all r-element subsets of $[n] := \{1, 2, \dots, n\}$ containing a fixed element, say 1, and adding to the family any collection of $\lfloor \frac{n-1}{r} \rfloor$ pairwise disjoint r-element subsets not containing 1. For $r = 3$, Füredi also gave an alternative lower bound construction using Steiner systems. An (n, r, t)-*Steiner system* $S(n, r, t)$ is an r-uniform hypergraph on $[n]$ in which every t-element subset of $[n]$ is contained in exactly one hyperedge. Füredi observed that if we replace every hyperedge in $S(n, 5, 2)$ by all its 3-element subsets then the resulting triple system has $\binom{n}{2}$ triples and contains no copy of $K_{2,2}^{(3)}$. This slightly improves the lower bound in (1) for $r = 3$ to $\binom{n}{2}$, for those n for which $S(n, 5, 2)$ exists. The upper bound in (1) was improved by Mubayi and Verstraëte [9] to $3\binom{n}{r-1} + O(n^{r-2})$. They obtain this bound by first showing $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)}) < 3\binom{n}{2} + 6n$, and then combining it with a simple reduction lemma. This was later improved to $f_3(n) \le \frac{13}{9}\binom{n}{2}$ by Pikhurko and Verstraëte [10].

Motivated by Füredi's work, Mubayi and Verstraëte [9] initiated the study of the general problem of determining $\mathrm{ex}(n, K_{2,t}^{(r)})$ for any $t \ge 2$. They showed that for any $t \ge 2$ and $n \ge 2t$,

$$\mathrm{ex}(n, K_{2,t}^{(3)}) < t^4 \binom{n}{2},$$

and that for infinitely many n, $\mathrm{ex}(n, K_{2,t}^{(3)}) \ge \frac{2t-1}{3}\binom{n}{2}$, where the lower bound is obtained by replacing each hyperedge in $S(n, 2t+1, 2)$ with all its 3-element subsets.

Mubayi and Verstraëte noted that $g(t) := \lim_{n\to\infty} \mathrm{ex}(n, K_{2,t}^{(3)})/\binom{n}{2}$ exists and raised the question of determining the growth rate of $g(t)$. Their results show that

$$\frac{2t-1}{3} \le g(t) \le t^4. \tag{2}$$

In this paper, we prove that as $t \to \infty$,

$$g(t) = \Theta(t^{1+o(1)}), \tag{3}$$

showing that their lower bound is close to the truth. More precisely, we prove the following.

Theorem 1. *For any $t \ge 2$, we have*

$$\mathrm{ex}(n, K_{2,t}^{(3)}) \le (15t \log t + 40t)\, n^2.$$

Notation. Given a hypergraph (or a graph) H, throughout the paper, we also denote the set of its edges by H. For example $|H|$ denotes the number of edges of H. Given two vertices x, y in a graph G, let $N_G(x, y)$ denote the common neighborhood of x and y in G. We drop the subscript G when the context is clear.

2. Proof of Theorem 1: $K_{2,t}^{(3)}$-free hypergraphs

We will use a special case of a well-known result of Erdős and Kleitman [2].

Lemma 1. *Let H be a 3-graph on $3n$ vertices. Then H contains a 3-partite 3-graph, with all parts of size n, and with at least $\frac{2}{9}|H|$ hyperedges.*

Let us define the sets $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$. Throughout the proof we define various 3-partite 3-graphs whose parts are A, B and C.

Suppose H is a $K_{2,t}^{(3)}$-free 3-partite 3-graph on $3n$ vertices with parts A, B and C. First let us show that it suffices to prove the following inequality.

$$|H| \leq (30t \log t + 80t)n^2. \tag{4}$$

It is easy to see that inequality (4) and Lemma 1 together imply that any $K_{2,t}^{(3)}$-free 3-graph on $3n$ vertices contains at most $\frac{9}{2}(30t \log t + 80t)n^2$ hyperedges, from which Theorem 1 would follow after replacing $3n$ by n.

In the remainder of the section, we will prove (4). Let us introduce the following notion of sparsity.

Definition 3 *(q-sparse and q-dense pairs).* Let q be a positive integer. Let G be a bipartite graph with parts X, Y. Let x, y be two different vertices such that $x, y \in X$ or $x, y \in Y$. Then we call $\{x, y\}$ a *q-dense* pair of G if $|N(x, y)| \geq q$. We call $\{x, y\}$ a *q-sparse* pair of G if $|N(x, y)| < q$ but x, y are still contained in a copy of $K_{2,q}$ in G. Note that it is possible that $\{x, y\}$ is neither q-sparse nor q-dense.

The following Procedure $\mathcal{P}(q)$ about making a bipartite graph $K_{2,q}$-free lies at the heart of the proof. (We think of q as the parameter of the Procedure $\mathcal{P}(q)$, that is changed throughout the proof.)

Procedure $\mathcal{P}(q)$: Making a bipartite graph $K_{2,q}$-free.

Input: A bipartite graph G with parts A and B.
  $\mathcal{G} \leftarrow G$, $\psi \leftarrow 1$.
  $F(x,y) \leftarrow \emptyset$, $D(x,y) \leftarrow \emptyset$ and $S(x,y) \leftarrow \emptyset$ for every $x, y \in A$ and $x, y \in B$.
  while $\psi = 1$ **do**
    $\psi \leftarrow 0$.
    Step 1:
    For each q-sparse pair $\{x, y\}$ of $\mathcal{G}$ such that $F(x,y) = \emptyset$, let $S(x,y)$ be the set of vertices spanned by the q-dense pairs of $\mathcal{G}$ that are contained in $N_{\mathcal{G}}(x,y)$. Let $F(x,y) \leftarrow \{ab \in \mathcal{G} \mid a \in \{x,y\}$ and $b \in S(x,y)\}$, and let $D(x,y)$ be a spanning forest of the graph formed by the dense pairs of $\mathcal{G}$ that are contained in $S(x,y)$.
    If there exists an edge $ab \in \mathcal{G}$ such that ab is contained in $F(x,y)$ for at least $q/2$ different pairs $\{x,y\}$ with $x, y \in A$ or for at least $q/2$ different pairs $\{x,y\}$ with $x, y \in B$,
    then $\mathcal{G} \leftarrow \mathcal{G} \setminus \{ab\}$ and $\psi \leftarrow 1$.
    Step 2:
    If there exists a set M of edges in $\mathcal{G}$ such that removing all of the edges of M from $\mathcal{G}$ would decrease the number of q-dense pairs by at least $|M|/2$,
    then $\mathcal{G} \leftarrow \mathcal{G} \setminus M$ and $\psi \leftarrow 1$.
  end while
  $G' \leftarrow \mathcal{G}$
  $F'(x,y) \leftarrow F(x,y)$ for every $x, y \in A$ and $x, y \in B$.
  $D'(x,y) \leftarrow D(x,y)$ for every $x, y \in A$ and $x, y \in B$.
  $S'(x,y) \leftarrow S(x,y)$ for every $x, y \in A$ and $x, y \in B$.
Output: The graph G' and the sets $F'(x,y), D'(x,y), S'(x,y)$ for all $x, y \in A$ and $x, y \in B$.

In the procedure $\mathcal{P}(q)$, initially for all the pairs $\{x, y\}$ (with $x, y \in A$ and $x, y \in B$) the sets $F(x, y)$, $D(x, y)$, $S(x, y)$ are set to be empty. Then as the edges are being deleted during the procedure, possibly, new q-sparse pairs $\{x, y\}$ are being created. When this happens, Step 1 redefines the sets $S(x, y), F(x, y), D(x, y)$ and gives them some non-empty values. (They get non-empty values due to the fact that $\{x, y\}$ is q-sparse, which implies that $\{x, y\}$ is contained in a copy of $K_{2,q}$, so there is at least one q-dense pair in the common neighborhood of x, y.) Therefore, these values stay unchanged throughout the rest of the procedure.

Notice that at the point $S(x, y)$ was redefined, the pair $\{x, y\}$ was q-sparse, so the number of common neighbors is less than q. Therefore, as $S(x, y)$ is a subset of the common neighborhood of x and y, we also have $|S(x, y)| < q$. Moreover, since $D(x, y)$ is defined as a spanning forest with the vertex set $S(x, y)$, we have $|D(x, y)| \leq |S(x, y)|$. Also, it easily follows from the definition of $F(x, y)$ that $|F(x, y)| = 2\,|S(x, y)|$. Finally, notice that $D(x, y)$ does not contain any isolated vertices, because its vertex set $S(x, y)$ spans all of its edges, by definition. Therefore, $|D(x, y)| \geq |S(x, y)|/2$. At the end of the procedure, the sets $F(x, y), D(x, y), S(x, y)$ are renamed as $F'(x, y), D'(x, y), S'(x, y)$. Note also that if a pair $\{x, y\}$ never becomes q-sparse in the process then $S'(x, y) = D'(x, y) = F'(x, y) = \emptyset$.

Observation 1. For every $x, y \in A$ and for every $x, y \in B$, we have

(1) $|S'(x, y)| < q$.
(2) $|D'(x, y)| \leq |S'(x, y)|$.
(3) $|F'(x, y)| = 2\,|S'(x, y)|$.
(4) $|D'(x, y)| \geq \frac{|S'(x,y)|}{2}$.

For convenience, throughout the paper we (informally) say that the sets $F'(x, y)$, $D'(x, y)$, $S'(x, y)$ are defined by applying Procedure $\mathcal{P}(q)$ to a graph G to obtain the graph G', instead of saying that the input to Procedure $\mathcal{P}(q)$ is G and the output is the graph G' and the sets $F'(x, y)$, $D'(x, y)$, $S'(x, y)$. Note that the output is not unique and may depend on the order in which edges were deleted when Procedure $\mathcal{P}(q)$ is applied to a graph G, but we just fix one such output and define G', $F'(x, y)$, $D'(x, y)$, $S'(x, y)$ with respect to that output.

Claim 1. *Let the sets $F'(x, y), D'(x, y), S'(x, y)$ (for $x, y \in A$ and for $x, y \in B$) be defined by applying Procedure $\mathcal{P}(q)$ to a bipartite graph G to obtain G'. Let $N(x, y)$ denote the set of common neighbors of vertices x, y in the graph G. Then*

$$\frac{|F'(x, y)|}{4} \leq |D'(x, y)| < q.$$

Moreover $|F'(x, y)| \leq 2\,|N(x, y)|$.

Proof. Combining the parts (3) and (4) of Observation 1, we have

$$|F'(x,y)|/4 \leq |D'(x,y)|.$$

Combining the parts (1) and (2) of Observation 1, we obtain

$$|D'(x,y)| < q,$$

proving the first part of the claim.

To prove the second part, notice that $S'(x,y)$ is a common neighborhood of x, y in some subgraph $\mathcal{G}$ of G, we have $|S'(x,y)| \leq |N(x,y)|$. Combining this with part (3) of Observation 1, we obtain $|F'(x,y)| \leq 2|N(x,y)|$, as required. □

Finally, let us note the following properties of the graph obtained after applying the procedure.

Observation 2. Let the sets $F'(x,y), D'(x,y), S'(x,y)$ (for $x, y \in A$ and $x, y \in B$) be defined by applying Procedure $\mathcal{P}(q)$ to a bipartite graph G to obtain G'. Then

1. Every edge ab in G' is contained in at most $q/2$ members of $\{F'(x,y) : x, y \in A\}$ and in at most $q/2$ members of $\{F'(x,y) : x, y \in B\}$.
2. For any set M of edges in G', removing the edges of M from G' decreases the number of q-dense pairs by less than $|M|/2$.

Definition 4. Let H be a 3-partite 3-graph with parts A, B and C.

For each $1 \leq i \leq n$, let $G_i[H](A,B)$ be the bipartite graph with parts A and B, whose edge set is $\{ab \mid a \in A, b \in B, abc_i \in E(H)\}$. The graphs $G_i[H](B,C)$ and $G_i[H](A,C)$ are defined similarly.

Definition 5 *(Applying Procedure $\mathcal{P}(q)$ to a hypergraph).* Let H be a 3-partite 3-graph with parts A, B and C. We define the hypergraph H' as follows:

For each $1 \leq i \leq n$, let $G'_i[H](A,B)$, $G'_i[H](B,C)$, $G'_i[H](A,C)$ be the graphs obtained by applying the procedure $\mathcal{P}(q)$ to the graphs $G_i[H](A,B)$, $G_i[H](B,C)$, $G_i[H](A,C)$ respectively.

For each edge ab which was removed from $G_i[H](A,B)$ by the procedure $\mathcal{P}(q)$ (i.e. $ab \in G_i[H](A,B) \setminus G'_i[H](A,B)$) we remove the hyperedge abc_i from $\mathcal{H}$ (it may have been removed already). Similarly for each edge bc (resp. ac) which was removed from $G_i[H](B,C)$ (resp. $G_i[H](A,C)$) by the procedure $\mathcal{P}(q)$ we remove the hyperedge a_ibc (resp. ab_ic) from $\mathcal{H}$. Let the resulting hypergraph be H'. More precisely, the edge-set of H' is

$$\{a_ib_jc_k \in H \mid a_ib_j \in G'_k[H](A,B),\ b_jc_k \in G'_i[H](B,C),\ a_ic_k \in G'_j[H](A,C)\}.$$

We say H' is obtained from H by applying the Procedure $\mathcal{P}(q)$.

Remark 1. Let H' be obtained by applying the Procedure $\mathcal{P}(q)$ to the hypergraph H. Then,

$$\begin{aligned}|H| - |H'| \leq &\sum_{1\leq i\leq n} \left(|G_i[H](A,B)| - |G'_i[H](A,B)|\right) \\ &+ \sum_{1\leq i\leq n} \left(|G_i[H](B,C)| - |G'_i[H](B,C)|\right) \\ &+ \sum_{1\leq i\leq n} \left(|G_i[H](A,C)| - |G'_i[H](A,C)|\right).\end{aligned}$$

Indeed, if $a_ib_jc_k \in H \setminus H'$ then it is easy to see that $a_ib_j \in G_k[H](A,B) \setminus G'_k[H](A,B)$ or $b_jc_k \in G_i[H](B,C) \setminus G'_i[H](B,C)$ or $a_ic_k \in G_j[H](A,C) \setminus G'_j[H](A,C)$.

Lemma 2. *Let $q \geq 2$ be an even integer and G be a bipartite graph with parts A and B. Suppose G' is the graph obtained by applying Procedure $\mathcal{P}(q)$ to G. Then G' is $K_{2,q}$-free.*

Proof. Let us define a *q-broom* of size k to be a set of q-sparse pairs $\{x_0, x_j\}$ (with $1 \leq j \leq k$), and a q-dense pair $\{y, z\}$ such that $\{y, z\}$ is contained in the common neighborhood of x_0, x_j for every $1 \leq j \leq k$. Note that either $\{x_0, x_1, \dots, x_k\} \subseteq A$ and $\{y, z\} \subseteq B$ or $\{x_0, x_1, \dots, x_k\} \subseteq B$ and $\{y, z\} \subseteq A$.

Claim 2. *There is no q-broom of size $q/2$ in G'.*

Proof. Suppose by contradiction that there is a set of q-sparse pairs $\{x_0, x_j\}$ (with $1 \leq j \leq q/2$), and a q-dense pair $\{y, z\}$ such that $\{y, z\}$ is contained in the common neighborhood of x_0 and x_j for every $1 \leq j \leq q/2$. Then the edge x_0y is contained in the sets $F'(x_0, x_j)$ for every $1 \leq j \leq q/2$, which contradicts Observation 2. □

Let us suppose for a contradiction (to Lemma 2) that G' contains a copy of $K_{2,q}$. Then G' contains at least one q-dense pair. Without loss of generality we may assume there is a q-dense pair $\{a, a_1\}$ in A. Suppose $\{a, a_j\}$ (for $1 \leq j \leq p$) are all the q-dense pairs of G' containing the vertex a. For each $1 \leq j \leq p$, let $B_j \subseteq B$ be the common neighborhood of a and a_j in G'. By definition, $|B_j| \geq q$ for $1 \leq j \leq p$.

Claim 3. *For any $J \subseteq \{1, 2, \dots, p\}$, we have $\left|\bigcup_{j\in J} B_j\right| > 2\,|J|$.*

Proof. Let us assume for contradiction that there exists a $J \subseteq \{1, 2, \dots, p\}$ such that $\left|\bigcup_{j\in J} B_j\right| \leq 2\,|J|$. Let G^* be obtained from G' by deleting all the edges from a to $\bigcup_{j\in J} B_j$. For each $j \in J$, the pair $\{a, a_j\}$ has no common neighbor in G^* since we have removed all the edges from a to B_j. Thus the pair $\{a, a_j\}$ is not q-dense in G^*. So in forming G^* from G' the number of q-dense pairs decreases by at least $|J|$, while the number of edges decreases by $|\bigcup_{j\in J} B_j| \leq 2|J|$ edges, contradicting Observation 2. □

Let $B' = \bigcup_{1 \le j \le p} B_j$. For each vertex $v \in B'$ and let

$$J(v) := \{j \mid v \in B_j\},$$
$$D(v) := \{\{v,u\} \mid \{v,u\} \text{ is } q\text{-dense in } G' \text{ and } \{v,u\} \subseteq B_j \text{ for some } j \in J(v)\}.$$

In the next two claims, we will prove two useful inequalities concerning $|J(v)|$ and $|D(v)|$.

Claim 4. *For each $v \in B'$, $|J(v)| > 2\,|D(v)|$.*

Proof. Suppose for contradiction that there is a vertex $v \in B'$ such that $|J(v)| \le 2\,|D(v)|$. Let us delete all the edges of the form va_j, $j \in J(v)$, from G' and let the resulting graph be G^*. Since we deleted $|J(v)|$ edges, by Observation 2, the number of q-dense pairs decreases by less than $|J(v)|/2 \le |D(v)|$. So there exists $\{v,u\} \in D(v)$ such that $\{v,u\}$ is (still) q-dense in G^*. That is, $|N^*(v,u)| \ge q$, where $N^*(v,u)$ denotes the common neighborhood of v and u in G^*. Clearly each pair of vertices in $N^*(v,u)$ is contained in a copy of $K_{2,q}$ in G^* (and hence in G').

For each pair of vertices in $N^*(v,u)$, since it is contained in a copy of $K_{2,q}$ in G', it is either q-sparse or q-dense in G'. Note that $a \in N^*(v,u)$. If all the pairs $\{a,x\}$ with $x \in N^*(v,u) \setminus \{a\}$ are q-sparse in G' then the set of these pairs together with $\{v,u\}$ is a q-broom of size at least $q-1 \ge q/2$ in G', which contradicts Claim 2. So there exists a vertex $x \in N^*(v,u) \setminus \{a\}$ such that $\{a,x\}$ is q-dense in G'. Since v is adjacent to both a and x, by the definition of $J(v)$, $x = a_j$ for some $j \in J(v)$. However, by definition, in forming G^* we have removed vx from G'. This contradicts $x \in N^*(v,u)$ and completes the proof. □

Claim 5.

$$\sum_{v \in B'} |D(v)| \ge \frac{1}{2} \sum_{1 \le j \le p} |B_j|.$$

Proof. Fix any j with $1 \le j \le p$. Since $\{a, a_j\}$ is q-dense in G', every pair $\{x,y\} \subseteq B_j$ is contained in some copy of $K_{2,q}$ and hence is either q-dense or q-sparse in G'. Let v be any vertex in B_j and let $S(v) = \{y \in B_j \mid \{v,y\} \text{ is } q\text{-sparse in } G'\}$. By definition, the set $\{\{v,y\} \mid y \in S(v)\}$ together with $\{a,a_j\}$ is a q-broom of size $|S(v)|$. By Claim 2, $|S(v)| \le q/2 - 1 \le |B_j|/2 - 1$. Since $|D(v)| + |S(v)| \ge |B_j| - 1$, we have

$$|D(v)| \ge \frac{1}{2}\,|B_j| \tag{5}$$

Note that (5) holds for every $j = 1, \ldots, p$ and every $v \in B_j$.

Let us define an auxiliary bipartite graph G_{aux} with the parts $\{1, 2, \ldots, p\}$, B' such that a vertex $j \in \{1, 2, \ldots, p\}$ is joined to a vertex $y \in B'$ if and only if $y \in B_j$. Let

J be an arbitrary subset of $\{1, 2, \ldots, p\}$. The neighborhood of J in G_{aux} is precisely $\bigcup_{j \in J} B_j$. By Claim 3, $\left|\bigcup_{j \in J} B_j\right| > 2|J| \geq |J|$. Since this holds for every $J \subseteq \{1, \ldots, p\}$, by Hall's theorem [5] there exist distinct vertices $w_j \in B_j$, for $j = 1, \ldots, p$. By (5), for every $j \in \{1, \ldots, p\}, |D(w_j)| \geq \frac{1}{2}|B_j|$. Hence

$$\sum_{v \in B'} |D(v)| \geq \sum_{1 \leq j \leq p} |D(w_j)| \geq \frac{1}{2} \sum_{1 \leq j \leq p} |B_j|. \quad \square$$

If we view $\{B_1, \ldots, B_p\}$ as a hypergraph on the vertex set B', then the degree of a vertex $v \in B'$ in it is precisely $|J(v)|$ and the degree sum formula yields

$$\sum_{v \in B'} |J(v)| = \sum_{1 \leq j \leq p} |B_j|. \tag{6}$$

Using Claim 4 and Claim 5 we have

$$\sum_{v \in B'} |J(v)| > \sum_{v \in B'} 2|D(v)| \geq 2 \sum_{1 \leq j \leq p} \frac{1}{2}|B_j| = \sum_{1 \leq j \leq p} |B_j|,$$

which contradicts (6). This completes proof of Lemma 2. $\square$

In the next subsection we will prove a general lemma about making an arbitrary hypergraph $K_{1,2,q}$-free (for any given value of q). This lemma is used several times in the following subsections.

2.1. Applying Procedure $\mathcal{P}(q)$ to an arbitrary hypergraph H

Let q be an even integer and let $q \geq t$. Let H be an arbitrary $K_{2,t}^{(3)}$-free 3-partite 3-graph with parts A, B and C. In this subsection we will prove the following lemma that estimates the number of edges removed from the graphs $G_i = G_i[H](A, B)$ for $1 \leq i \leq n$, when the Procedure $\mathcal{P}(q)$ is applied to them. This lemma together with Remark 1 will allow us to estimate the number of edges removed from H when the Procedure $\mathcal{P}(q)$ is applied to it.

Throughout this subsection, $N_i(x, y)$ denotes the set of common neighbors of the vertices x, y in the graph G_i.

Lemma 3. *Let $q \geq t$ be an even integer. Let H be an arbitrary $K_{2,t}^{(3)}$-free 3-partite 3-graph with parts A, B and C. Let $G_i = G_i[H](A, B)$ for $1 \leq i \leq n$. For each $1 \leq i \leq n$ and any $x, y \in A$ or $x, y \in B$, let $F_i'(x, y)$ be defined by applying the procedure $\mathcal{P}(q)$ to G_i and let the resulting graph be G_i'. Then,*

$$\sum_{1 \leq i \leq n} |G_i \setminus G_i'| < \frac{2}{q} \left(\sum_{u,v \in A} \sum_{1 \leq i \leq n} |F_i'(u, v)| + \sum_{u,v \in B} \sum_{1 \leq i \leq n} |F_i'(u, v)| \right) + 2tn^2.$$

Proof of Lemma 3. First let us prove the following claim.

Claim 6. *Let $u, v \in A$ or $u, v \in B$. Then $\{u, v\}$ is q-dense in less than t of the graphs G_i, $1 \le i \le n$.*

Proof. Without loss of generality, suppose that $u, v \in A$. Suppose for contradiction that $\{u, v\}$ is q-dense in t of the graphs G_i, $1 \le i \le n$. Without loss of generality suppose $\{u, v\}$ is q-dense in $G_1, \dots, G_t$. Then $|N_i(u, v)| \ge q \ge t$ for $i = 1, \dots, t$. Therefore, we can greedily choose t distinct vertices $y_1, \dots, y_t$ such that for each $i \in [t], y_i \in N_i(u, v)$. For each $i \in [t]$, since $y_i \in N_i(u, v)$ we have $uy_ic_i, vy_ic_i \in E(H)$. However, the set of hyperedges $\{uy_ic_i, vy_ic_i \in E(H) \mid 1 \le i \le t\}$ forms a copy of $K_{2,t}^{(3)}$ in H, a contradiction. □

Note that when procedure $\mathcal{P}(q)$ is applied to G_i (to obtain G_i'), Step 1 and Step 2 may be applied several times (and each time one of these steps is applied it may delete an edge of G_i).

For each $i \in [n]$, let m_i denote the number of q-dense pairs of G_i. By Claim 6, we know that each pair $\{u, v\}$ with $u, v \in A$ or $u, v \in B$, is q-dense in less than t different graphs G_i (for $1 \le i \le n$). Therefore,

$$\sum_{1 \le i \le n} m_i \le \sum_{u,v \in A} (t-1) + \sum_{u,v \in B} (t-1) = 2\binom{n}{2}(t-1). \tag{7}$$

For each $i \in [n]$, let α_i denote the total number of edges that were removed by Step 1 when procedure $\mathcal{P}(q)$ is applied to G_i and β_i be the number of edges removed by Step 2 when procedure $\mathcal{P}(q)$ is applied to G_i. Then $\alpha_i + \beta_i = |G_i \setminus G_i'|$, so $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n |G_i \setminus G_i'|$.

First, we bound $\sum_{i=1}^n \beta_i$. Let $i \in [n]$. Observe that whenever a set M of edges were removed by Step 2 of Procedure $\mathcal{P}(q)$ applied to G_i, the number of q-dense pairs decreased by at least $|M|/2$. Hence $\beta_i \le 2m_i$. So summing up over all $1 \le i \le n$, and using (7), we get

$$\sum_{1 \le i \le n} \beta_i \le 2 \sum_{1 \le i \le n} m_i \le 2n(n-1)(t-1) < 2tn^2. \tag{8}$$

Next, we bound $\sum_{i=1}^n \alpha_i$. Let $i \in [n]$. If an edge xy were removed from G_i by Step 1 of the procedure $\mathcal{P}(q)$ then there are vertices $z_1, z_2, \dots, z_{q/2}$ such that $xy \in F_i'(x, z_j)$ for every $j \in \{1, 2, \dots, q/2\}$ or $xy \in F_i'(y, z_j)$ for every $j \in \{1, 2, \dots, q/2\}$. So

$$\alpha_i \le \frac{1}{q/2} \left(\sum_{u,v \in A} |F_i'(u, v)| + \sum_{u,v \in B} |F_i'(u, v)| \right).$$

Therefore,

$$\sum_{1\le i\le n}\alpha_i \le \frac{2}{q}\left(\sum_{1\le i\le n}\sum_{u,v\in A}|F_i'(u,v)| + \sum_{1\le i\le n}\sum_{u,v\in B}|F_i'(u,v)|\right).$$

This is equivalent to the following.

$$\sum_{1\le i\le n}\alpha_i \le \frac{2}{q}\left(\sum_{u,v\in A}\sum_{1\le i\le n}|F_i'(u,v)| + \sum_{u,v\in B}\sum_{1\le i\le n}|F_i'(u,v)|\right). \tag{9}$$

Combining this inequality with (8) completes the proof of Lemma 3. □

2.2. The overall plan

Let us define the sequence $q_0, q_1, \dots, q_k$ as follows. Let $q_0 = 2^l$ where l is an integer such that $q_0 = 2^l \le t^2 < 2^{l+1} = 2q_0$. For each $1 \le j \le k$, let $q_j = \frac{q_{j-1}}{2}$ and $q_k \ge t > \frac{q_k}{2}$. Clearly $\frac{q_0}{q_k} = 2^k$, moreover

$$2^k = \frac{q_0}{q_k} \le \frac{t^2}{t} = t.$$

So we have

$$k \le \log t. \tag{10}$$

Now we apply the procedure $\mathcal{P}(q_0)$ to the hypergraph H (recall Definition 5) to obtain a $K_{1,2,q_0}$-free hypergraph H_0. For each $0 \le j < k$ we obtain $K_{1,2,q_{j+1}}$-free hypergraph H_{j+1} by applying the procedure $\mathcal{P}(q_{j+1})$ to the hypergraph H_j.

This way, in the end we will get a $K_{1,2,q_k}$-free hypergraph H_k. In the following section, we will upper bound $|H| - |H_0|$. Then in the next section, using the information that H_j is $K_{1,2,q_j}$-free, we will upper bound $|H_{j+1}| - |H_j|$ for each $0 \le j < k$. Then we sum up these bounds to upper bound the total number of deleted edges (i.e., $|H| - |H_k|$) from H to obtain H_k. Finally, we bound the size of H_k, which will provide us the desired bound on the size of H.

2.3. Making H $K_{1,2,q_0}$-free

First, we are going to prove an auxiliary lemma that is similar to Lemma A.4 of [9]. In an edge-colored multigraph G, an *s-frame* is a collection of s edges all of different colors such that it is possible to pick one endpoint from each edge with all the selected endpoints being distinct.

Lemma 4. *Let G be an edge-colored multigraph with e edges such that each edge has multiplicity at most p and each color class has size at most q. If G contains no t-frame then $|G| \le \binom{t-1}{2}p + tq$.*

Proof. Consider a maximum frame S, say with edges $e_1,\dots,e_s$ such that for every $i \in \{1,2,\dots,s\}$, e_i has color i and that there exist $x_1 \in e_1, x_2 \in e_2, \dots, x_s \in e_s$ with $x_1,\dots,x_s$ being distinct. By our assumption, $s \le t-1$. Let f be any edge with a color not in $[s]$. Then both vertices of f must be in $\{x_1,\dots,x_s\}$, otherwise $e_1,\dots,e_s,f$ give a larger frame, a contradiction. On the other hand, each edge with both of its vertices in $\{x_1,\dots,x_s\}$ has multiplicity at most p. Hence there are at most $\binom{s}{2}p$ edges with colors not in $\{1,2,\dots,s\}$. The number of edges with color in $\{1,2,\dots,s\}$ is at most sq by our assumption. So $|G| \le \binom{s}{2}p + sq \le \binom{t-1}{2}p + tq$. □

Let us recall that H is 3 partite $K_{2,t}^{(3)}$-free hypergraph with A,B,C. For convenience we denote $G_i = G_i[H](A,B)$ where $1 \le i \le n$. For each $1 \le i \le n$ and any $x,y \in A$ or $x,y \in B$, let $F_i'(x,y)$, $D_i'(x,y)$ and $S_i'(x,y)$ be defined by applying the procedure $\mathcal{P}(q_0)$ on G_i and let the obtained graph be G_i'.

First, observe that $t^2/2 < q_0 \le t^2$ according to our definition.

Claim 7. *Let $u,v \in A$ or $u,v \in B$. Then $\sum_{1\le i\le n} |F_i'(u,v)| \le 6t^3$.*

Proof. Let D^* be an edge-colored multigraph in which a pair of vertices e is an edge of color $i \in [n]$ whenever e is an edge of $D_i'(u,v)$. The number of edges of color i in D^* is $|D_i'(u,v)|$. By Claim 1 we have $|D_i'(u,v)| < q_0$. Hence the number of edges in each color class of D^* is less than q_0.

Let xy be an arbitrary edge of D^* and let $I = \{i \in [n] \mid xy \in D_i'(u,v)\}$. For each $i \in I$, the pair $\{x,y\}$ is q_0-dense in G_i by the definition of $D_i'(u,v)$. Therefore, by Claim 6, we have $|I| < t$. So xy has multiplicity less than t in D^*. Since xy is arbitrary, the multiplicity of each edge of D^* is less than t.

Next, observe that D^* contains no t-frame. Indeed, otherwise without loss of generality we may assume that D^* contains t edges $x_1y_1,\dots,x_ty_t$, where x_iy_i has color i for each $i \in [t]$ and $y_1,\dots,y_t$ are distinct. For each $i \in [t]$ since $x_iy_i \in D_i'(u,v)$, in particular $y_i \in N_i(u,v)$ (where $N_i(u,v)$ denotes the common neighborhood of u and v in G_i), which means that $uy_ic_i, vy_ic_i \in H$. But now, $\{uy_ic_i, vy_ic_i \mid i \in [t]\}$ forms a copy of $K_{2,t}^{(3)}$, contradicting H being $K_{2,t}^{(3)}$-free.

Therefore, applying Lemma 4, we have $|D^*| \le \binom{t-1}{2}t + tq_0$. By Claim 1, we have

$$\frac{|F_i'(u,v)|}{4} \le |D_i'(u,v)|.$$

So

$$\sum_{1\le i\le n} \frac{|F_i'(u,v)|}{4} \le \sum_{1\le i\le n} |D_i'(u,v)| = |D^*| \le \binom{t-1}{2}t + tq_0 < \frac{3}{2}t^3,$$

which proves the claim. □

By Lemma 3 we have

$$\sum_{1\le i\le n} |G_i \setminus G_i'| < \frac{2}{q_0}\left(\sum_{u,v\in A}\sum_{1\le i\le n}|F_i'(u,v)| + \sum_{u,v\in B}\sum_{1\le i\le n}|F_i'(u,v)|\right) + 2tn^2.$$

Combining it with Claim 7 we get

$$\sum_{1\le i\le n} |G_i \setminus G_i'| < \frac{2}{q_0}\left(\sum_{u,v\in A} 6t^3 + \sum_{u,v\in B} 6t^3\right) + 2tn^2.$$

Therefore, as $q_0 > t^2/2$, we have

$$\sum_{1\le i\le n} |G_i \setminus G_i'| < \frac{4}{t^2}\left(12t^3\binom{n}{2}\right) + 2tn^2 < 26tn^2.$$

So,

$$\sum_{1\le i\le n} |G_i \setminus G_i'| = \sum_{1\le i\le n} |G_i[H](A,B) \setminus G_i'[H](A,B)| < 26tn^2.$$

By symmetry, using the same arguments, we have

$$\sum_{1\le i\le n} |G_i[H](B,C) \setminus G_i'[H](B,C)| < 26tn^2,$$

and

$$\sum_{1\le i\le n} |G_i[H](A,C) \setminus G_i'[H](A,C)| < 26tn^2.$$

Therefore, by Remark 1, we have

$$|H| - |H_0| < 78tn^2. \tag{11}$$

2.4. Making a $K_{1,2,q_j}$-free hypergraph $K_{1,2,q_{j+1}}$-free

In this subsection, we fix a j with $0 \le j < k$. Recall that H_j is $K_{1,2,q_j}$-free, and H_{j+1} is obtained by applying the $\mathcal{P}(q_{j+1})$ to H_j. Our goal in this subsection is to estimate $|H_j|-|H_{j+1}|$. The key difference between arguments in this subsection and in the previous subsection is that now in addition to H_j being $K_{2,t}^{(3)}$-free we can also utilize the fact that H_j is $K_{1,2,q_j}$-free. In particular, this extra condition leads to Claim 8, which improves upon Claim 7.

For convenience of notation, in this subsection, let $G_i = G_i[H_j](A,B)$ for each $1 \le i \le n$. For every $1 \le i \le n$ and every $u,v \in A$ or $u,v \in B$ let the sets $F_i'(u,v)$ and $D_i'(u,v)$ be defined by applying the procedure $\mathcal{P}(q_{j+1})$ to the graph G_i, to obtain the graph G_i'.

Claim 8. *Let $u,v \in A$ or $u,v \in B$. Then $\sum_{1\le i\le n} |F_i'(u,v)| < 2q_j t$.*

Proof. For each $i \in [n]$ we denote the set of common neighbors of u,v in G_i as $N_i(x,y)$. For each $i \in [n]$, since H_j is $K_{1,2,q_j}$-free, G_i is K_{2,q_j}-free and so $|N_i(u,v)| < q_j$.

Without loss of generality let us assume $u,v \in A$. For each vertex w of B, let $I_w = \{i \in \{1,2,\dots,n\} \mid w \in N_i(u,v)\}$. We claim that $|I_w| < q_j$. Indeed, for each $i \in I_w$, we have $uwc_i, vwc_i \in H_j$. So the set of hyperedges $\{uwc_i, vwc_i \mid i \in I_w\}$ form a copy of $K_{1,2,|I_w|}$ in H_j. Thus if $|I_w| \ge q_j$, then H_j contains a copy of $K_{1,2,q_j}$, a contradiction. Therefore, $|I_w| < q_j$, as desired.

Consider an auxiliary bipartite graph G_{AUX} with parts B and $[n]$ where the vertex $i \in [n]$ is adjacent to $b \in B$ in G_{AUX} if and only if $b \in N_i(u,v)$. Then by the discussion in the previous paragraph, each vertex $w \in B$ has degree $|I_w| < q_j$, and each vertex $i \in [n]$ has degree $|N_i(u,v)| < q_j$. In other words, the maximum degree in G_{AUX} is less than q_j.

We claim that G_{AUX} does not contain a matching of size t. Indeed, suppose for a contradiction that the edges $i_1b_{i_1}, i_2b_{i_2}, \dots, i_tb_{i_t}$ (i.e., $b_{i_l} \in N_{i_l}(u,v)$ for $1 \le l \le t$) form a matching of size t in G_{AUX}. Then the set of hyperedges $ub_{i_l}c_{i_l}, vb_{i_l}c_{i_l}$, $1 \le l \le t$, form a copy of $K_{2,t}^{(3)}$ in H_j, a contradiction, as desired.

Since G_{AUX} does not contain a matching of size t, by the König-Egerváry theorem it has a vertex cover of size less than t. This fact combined with the fact that the maximum degree of G_{AUX} is less than q_j, implies that the number of edges of G_{AUX} is less than $q_j t$. On the other hand, the number of edges in G_{AUX} is $\sum_{i\in[n]} |N_i(u,v)|$. Therefore, $\sum_{i\in[n]} |N_i(u,v)| < q_j t$. This, combined with the fact that for each $i \in [n]$, $|N_i(u,v)| \ge |F_i'(u,v)|/2$ (see Claim 1), completes the proof of the lemma. □

By Lemma 3, we have

$$\sum_{1\le i\le n} |G_i \setminus G_i'| \le \frac{2}{q_{j+1}} \left(\sum_{u,v,\in A} \sum_{1\le i\le n} |F_i'(u,v)| + \sum_{u,v,\in B} \sum_{1\le i\le n} |F_i'(u,v)| \right) + 2tn^2.$$

Now using Claim 8, we have

$$\sum_{1\le i\le n} |G_i \setminus G_i'| \le \frac{8q_jt}{q_{j+1}} \binom{n}{2} + 2tn^2 < \frac{4tq_j}{q_{j+1}} n^2 + 2tn^2.$$

Since $q_{j+1} = q_j/2$, we have

$$\sum_{1\le i\le n} |G_i \setminus G_i'| < 8tn^2 + 2tn^2 = 10tn^2.$$

So,

$$\sum_{1\le i\le n} |G_i \setminus G_i'| = \sum_{1\le i\le n} |G_i[H_j](A,B) \setminus G_i'[H_j](A,B)| < 10tn^2.$$

By symmetry, using the same arguments, we have

$$\sum_{1\le i\le n} |G_i[H_j](B,C) \setminus G_i'[H_j](B,C)| < 10tn^2,$$

and

$$\sum_{1\le i\le n} |G_i[H_j](A,C) \setminus G_i'[H_j](A,C)| < 10tn^2.$$

Therefore, by Remark 1, we have

$$|H_j| - |H_{j+1}| < 30tn^2. \tag{12}$$

2.5. Putting it all together

By (11) and (12) we have

$$|H| - |H_k| = |H| - |H_0| + \sum_{0\le j<k} (|H_j| - |H_{j+1}|) < 78tn^2 + k(30tn^2).$$

By (10) we have $k \le \log t$, so we obtain,

$$|H| - |H_k| < 78tn^2 + 30t\log tn^2. \tag{13}$$

Notice that H_k is $K_{1,2,q_k}$-free and $q_k < 2t$. Therefore H_k is $K_{1,2,2t}$-free. Moreover, we know that the hypergraph H_k is 3-partite and $K_{2,t}^{(3)}$-free with parts A, B, C (as it is a subhypergraph of H). Now we bound the size of H_k.

Claim 9. *We have* $|H_k| \le 2tn^2$.

Proof. Suppose for a contradiction that $|H_k| > 2tn^2$. For any pair $\{a, b\}$ of vertices with $a \in A$ and $b \in B$, let $\mathrm{codeg}(a, b)$ denote the number of hyperedges of H_k containing the pair $\{a, b\}$. Then the number of copies of $K_{2,1,1}$ in H_k of the form $\{abc, a'bc\}$ where $a, a' \in A$, $b \in B$, $c \in C$ is

$$\sum_{\substack{b,c\\ b\in B, c\in C}} \binom{\mathrm{codeg}(b,c)}{2}.$$

As the average codegree (over all the pairs $b \in B, c \in C$) is more than $2t$, by convexity, this expression is more than

$$\binom{2t}{2} n^2 > (2t-1)^2 \binom{n}{2}.$$

This means there exist a pair $a, a' \in A$ and a set of $(2t-1)^2+1 > (t-1)(2t-1)+1$ pairs $S := \{bc \mid b \in B, c \in C\}$ such that $abc, a'bc \in E(H_k)$ whenever $bc \in S$. Let G_{AUX} be a bipartite graph whose edges are elements of S. Since G_{AUX} has $|S| \geq (t-1)(2t-1)+1$ edges, it either contains a matching M with t edges or a vertex v of degree $2t$ (see Lemma A.3 in [9] or the last paragraph of our proof of Claim 8 for a proof). In the former case, the set of all hyperedges of the form $abc, a'bc$ with $bc \in M$, form a copy of $K_{2,t}^{(3)}$ in H_k, a contradiction. In the latter case, let $u_1, u_2, \ldots, u_{2t}$ be the neighbors of v in G_{AUX}. Then the set of hyperedges $\{avu_i, a'vu_i \mid 1 \leq i \leq 2t\}$ form a copy of $K_{1,2,2t}$ in H_k, a contradiction again. This completes the proof of the claim. □

Combining (13) with Claim 9, we have $|H| \leq 80tn^2 + 30t \log tn^2$, thus proving (4), which implies Theorem 1, as desired.

3. Concluding remarks

Recall that given a bipartite graph G with an ordered bipartition (X, Y), where $Y = \{y_1, \ldots, y_m\}$, $G_{X,Y}^{(r)}$ is the r-graph with vertex set $(X \cup Y) \cup (\bigcup_{i=1}^m Y_i)$ and edge set $\bigcup_{i=1}^m \{e \cup Y_i : e \in E(G), y_i \in e\}$, where $Y_1, \ldots, Y_m$ are disjoint $(r-2)$-sets that are disjoint from $X \cup Y$. The proof of Theorem 1.4 in [9] implies the following.

Proposition 1. *Let $n, r \geq 3$ be integers and G a bipartite graph with an ordered bipartition (X, Y). There exists a constant c_r depending only on r such that*

$$\mathrm{ex}(n, G_{X,Y}^{(r)}) \leq c_r n^{r-3} \cdot \mathrm{ex}(n, G_{X,Y}^{(3)}).$$

Thus, by Theorem 1 and Proposition 1, for all $r \geq 4$, we have $\mathrm{ex}(n, K_{2,t}^{(r)}) \leq c_r t \log t \binom{n}{r-1}$ for some constant c_r, depending only on r. On the other hand, taking the family of all r-element subsets of $[n]$ containing a fixed element shows that $\mathrm{ex}(n, K_{2,t}^{(r)}) \geq \binom{n-1}{r-1}$. Recall that in the $r = 3$ case, a better lower bound of $\Omega(t\binom{n}{2})$ was shown by Mubayi and Verstraëte [9]. For $r = 4$, we are able to improve the lower bound to $\Omega(t\binom{n}{3})$ as follows.

Proposition 2. *We have*

$$\mathrm{ex}(n, K_{2,t}^{(4)}) \geq (1 + o(1)) \frac{t-1}{8} n^3.$$

Proof. (Sketch.) Consider a $K_{2,t}$-free graph G with $(1+o(1))\frac{\sqrt{t-1}}{2}n^{3/2}$ edges where each vertex has degree $(1+o(1))\sqrt{(t-1)}\sqrt{n}$. (Such a graph exists by a construction of Füredi [3].) Let us a define a 4-graph $H = \{abcd \mid ab, cd \in G \text{ and } ac, ad, bc, bd \notin G\}$. In other words, let the edges of H be the vertex sets of induced 2-matchings in G. Via standard counting, it is easy to show that $|H| = (1+o(1))\frac{t-1}{8}n^3$. It remains to show H is $K_{2,t}^{(4)}$-free.

Claim 10. *If $axyz, bxyz \in H$, then there is a vertex $c \in \{x, y, z\}$ such that $ac, bc \in G$.*

Proof. By our assumption, $\{a, x, y, z\}$ and $\{b, x, y, z\}$ both induce a 2-matching in G. Without loss of generality, suppose $ax, yz \in G$. If $bx \in G$ then we are done. Otherwise, we have $by, xz \in G$ or $bz, xy \in G$, both contradicting $\{ax, yz\}$ being an induced matching in G. □

Suppose for contradiction that H has a copy of $K_{2,t}^{(4)}$ whose edgeset is $\{ax_iy_iz_i, bx_iy_iz_i \mid 1 \le i \le t\}$. By Claim 10, for each $1 \le i \le t$, there exists a vertex $w_i \in \{x_i, y_i, z_i\}$ such that $aw_i, bw_i \in G$. This yields a copy of $K_{2,t}$ in G, a contradiction. □

For $r \ge 5$, we do not yet have a lower bound that is asymptotically larger than $\binom{n-1}{r-1}$. It would be interesting to narrow the gap between the lower and upper bounds on $\mathrm{ex}(n, K_{2,t}^{(r)})$.

It will be interesting to have a systematic study of the function $\mathrm{ex}(n, G_{X,Y}^{(r)})$. Mubayi and Verstraëte [9] showed that $\mathrm{ex}(n, K_{s,t}^{(3)}) = O(n^{3-1/s})$ and that if $t > (s-1)! > 0$ then $\mathrm{ex}(n, K_{s,t}^{(3)}) = \Omega(n^{3-2/s})$ and speculated that $n^{3-2/s}$ is the correct order of magnitude. The case when G is a tree is studied in [4], where the problem considered there is slightly more general. The case when G is an even cycle has also been studied. Let $C_{2t}^{(r)}$ denote $G_{X,Y}^{(r)}$ where G is the even cycle C_{2t} of length $2t$. It was shown by Jiang and Liu [6] that $c_1 t\binom{n}{r-1} \le \mathrm{ex}(n, C_{2t}^{(r)}) \le c_2 t^5\binom{n}{r-1}$, for some positive constants c_1, c_2 depending on r. Using results in this paper and new ideas, we are able to narrow the gap to $c_1 t\binom{n}{r-1} \le \mathrm{ex}(n, C_{2t}^{(r)}) \le c_2 t^2 \log t\binom{n}{r-1}$, for some positive constants c_1, c_2 depending on r. We would like to postpone this and other results on the topic for a future paper.

Finally, motivated by results on $K_{2,t}^{(r)}$ and $C_{2t}^{(r)}$, we pose the following question.

Question 1. *Let $r \ge 3$. Let $\mathcal{G}$ be the family of bipartite graphs G with an ordered bipartition (X, Y) in which every vertex in Y has degree at most 2 in G. Is it true that $\forall G \in \mathcal{G}$ there is a constant c depending on G such that $\mathrm{ex}(n, G_{X,Y}^{(r)}) \le c\binom{n}{r-1}$?*

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