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TURÁN NUMBERS OF BIPARTITE SUBDIVISIONS*

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Abstract. Given a graph H, the *Turán number* $\mathrm{ex}(n,H)$ is the largest number of edges in an H-free graph on n vertices. We make progress on a recent conjecture of Conlon, Janzer, and Lee [*More on the Extremal Number of Subdivisions*, arXiv:1903.10631v1, 2019] on the Turán numbers of bipartite graphs, which in turn yields further progress on a conjecture of Erdős and Simonovits [*Combinatorica*, 1 (1981), pp. 25–42]. Let $s,t,k \geq 2$ be integers. Let $K_{s,t}^k$ denote the graph obtained from the complete bipartite graph $K_{s,t}$ by replacing each edge uv in it with a path of length k between u and v such that the st replacing paths are internally disjoint. It follows from a general theorem of Bukh and Conlon [J. Eur. Math. Soc. (JEMS), 20 (2018), pp. 1747–1757] that $\mathrm{ex}(n,K_{s,t}^k) = \Omega(n^{1+\frac{1}{k}-\frac{1}{sk}})$. Conlon, Janzer, and Lee recently conjectured that for any integers $s,t,k \geq 2$, $\mathrm{ex}(n,K_{s,t}^k) = O(n^{1+\frac{1}{k}-\frac{1}{sk}})$. Among many other things, they settled the $k=2$ case of their conjecture. As the main result of this paper, we prove their conjecture for $k=3,4$. Our main results also yield infinitely many new so-called *Turán exponents*: rationals $r \in (1,2)$ for which there exists a bipartite graph H with $\mathrm{ex}(n,H) = \Theta(n^r)$, adding to the lists recently obtained by Jiang, Ma, and Yepremyan [*On Turán Exponents of Bipartite Graphs*, arXiv:1806.02838, 2018], by Kang, Kim, and Liu [*On the Rational Turán Exponent Conjecture*, arXiv:1811.06916, 2018], and by Conlon, Janzer, and Lee. Our method builds on an extension of the Conlon–Janzer–Lee method. We also note that the extended method also gives a weaker version of the Conlon–Janzer–Lee conjecture for all $k \geq 2$.

Key words. Turán number, Turán exponent, extremal function, subdivision

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1. Introduction.

Given a family $\mathcal{H}$ of graphs, the *Turán number* $\mathrm{ex}(n,\mathcal{H})$ is the largest number of edges in an n-vertex graph that does not contain any member of $\mathcal{H}$. If $\mathcal{H}$ consists of a single graph H, we write $\mathrm{ex}(n,H)$ for $\mathrm{ex}(n,\{H\})$. Let $p = \min\{\chi(H)-1 : H \in \mathcal{H}\}$, where $\chi(H)$ denotes the chromatic number of H. The celebrated Erdős–Stone–Simonovits theorem [9, 11] asserts that $\mathrm{ex}(n,\mathcal{H}) = (1-\frac{1}{p}+o(1))\binom{n}{2}$. This determines the function for all families that do not contain a bipartite member. When $\mathcal{H}$ contains a bipartite graph, the problem is generally wide-open, with many intriguing conjectures. See [16] for a recent survey and [10, 12, 13, 14, 15, 25, 27] among others for some earlier work. One of these, known as the *Turán exponent conjecture*, was made by Erdős and Simonovits (see [8]) and asserts that for any rational $r \in (1,2)$ there exists a bipartite graph H such that $\mathrm{ex}(n,H) = \Theta(n^r)$. We call a rational r for which the Erdős–Simonovits conjecture holds a *Turán exponent*. In a recent breakthrough, Bukh and Conlon [2] proved that for any rational number $r \in (1,2)$ there exists a finite family $\mathcal{H}$ of graphs such that $\mathrm{ex}(n,\mathcal{H}) = \Theta(n^r)$. On the other hand, the original conjecture of Erdős and Simonovits concerning single bipartite graphs is still generally open. Until recently, it was only known to be true for $r = 1+1/k$ and $r = 2-1/k$, where $k \geq 2$ is a positive integer. Recently, there has been a lot of work done on the conjecture, by Jiang, Ma, and Yepremyan

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[21], by Kang, Kim, Liu [24], and by Conlon, Janzer, and Lee [6]. For more detailed discussions on recent works on the Erdős–Simonovits conjecture, the reader is referred to [2, 21, 24, 6]. See [19] for a rencent work on a related problem.

A recent focal point on the Erdős–Simonovits conjecture, with motivations from other problems as well, concerns the Turán number of so-called subdivisions of graphs. Given a graph H and an integer $k \geq 2$, let H^k denote the graph obtained by replacing each edge uv of H with a path of length k between u and v so that the $e(H)$ replacing paths are internally vertex disjoint. The Turán number of H^k is studied in [20] and [22], based on earlier work in [26]. Recently, significant progress on the problem have been made in [7], [17], and [6]. Let $s, t, k \geq 2$ be integers. As usual, let $K_{s,t}$ denote the complete bipartite graph with part sizes s and t. Let $K_{s,t}^k = (K_{s,t})^k$. It follows from the above-mentioned breakthrough work of Bukh and Conlon [2] that $\mathrm{ex}(n, K_{s,t}^k) = \Omega(n^{1+\frac{1}{k}-\frac{1}{sk}})$ for sufficiently large t. Conlon, Janzer, and Lee [6] recently made the following conjecture on a matching upper bound.

Conjecture 1.1 (see [6]). *For any integers $s, t, k \geq 2$, $\mathrm{ex}(n, K_{s,t}^k) = O(n^{1+\frac{1}{k}-\frac{1}{sk}})$.*

In [6], among many other things, Conlon, Janzer, and Lee settled the $k = 2$ case of Conjecture 1.1, showing that $\mathrm{ex}(n, K_{s,t}^2) = O(n^{\frac{3}{2}-\frac{1}{2s}})$. In this paper, we prove their conjecture for $k = 3, 4$.

Theorem 1.1. *For any integers $s, t \geq 2$ and $k \in \{3, 4\}$, $\mathrm{ex}(n, K_{s,t}^k) = O(n^{1+\frac{1}{k}-\frac{1}{sk}})$.*

We remark that our theorem together with the theorem of Bukh and Conlon also yields infinitely many new *Turán exponents*: namely, those of the form $1 + \frac{1}{k} - \frac{1}{sk}$, where $s \geq 2$ is any integer and $k \in \{3, 4\}$. The majority of the rest of the paper is devoted to the proof of our main result: Theorem 1.1. We then conclude with some observations in the concluding remarks.

2. Terminologies, preliminary lemmas, and earlier results. As is often the case in the study of bipartite Turán problems, our problem may be reduced to the setting in which the host graph is almost regular. Specifically, given a positive integer K, we say that a graph G is K*-almost-regular* if $\Delta(G) \leq K \cdot \delta(G)$.

The following lemma can be found in [22], which is a slight adaption of the regularization lemma of Erdős and Simonovits [10]. Another recent adaption of this can be found in [6].

Lemma 2.1 (see [22, Proposition 2.7]). *Let $0 < \epsilon < 1$ and $c \geq 1$. There exists $n_0 = n_0(\epsilon) > 0$ such that the following holds for all $n \geq n_0$. If G is a graph on n vertices with $e(G) \geq cn^{1+\epsilon}$, then G contains a K-almost-regular subgraph G' on $m \geq n^{\frac{\epsilon-\epsilon^2}{2+2\epsilon}}$ vertices such that $e(G') \geq \frac{2c}{5} m^{1+\epsilon}$ and $K = 20 \cdot 2^{\frac{1}{\epsilon^2}+1}$.*

For most of the rest of the paper we will always assume our host graph G to be almost regular. Then in the main proof we apply Lemma 2.1 on general host graphs.

The following two definitions are due to Conlon, Janzer, and Lee [6]. To make our presentation consistent with the rest of our paper, we present their terminologies and results using our terminologies.

Definition 2.2 (Definition 6.2 of [6]). *Let L be an integer. Define the function $f(\ell, L)$ for $0 \leq \ell \leq k$ recursively by setting $f(0, L) = 1$, $f(1, L) = L$, and for $\ell \geq 2$,*

$$f(\ell, L) = 1 + f(\ell - 1, L)^{16}(\ell - 1)^2 \max_{1 \leq i \leq \ell - 1} f(i, L) f(\ell - i, L).$$

Definition 2.3. *We recursively define j-admissible and j-light paths in a graph G. Any edge in G is both 1-admissible and 1-light. For $j \geq 2$, a path in G is j-admissible if it has length j and its every subpath of length $\ell < j$ is ℓ-light. A path* *from x to y in G is j-light if it is j-admissible and the number of j-admissible paths in G between x and y is at most $f(j, L)$. A path in G is j-critical if it is j-admissible but not j-light.*

When the length is clear, we often drop the prefixes in Definition 2.3. The following lemma is implied by Lemma 6.8 and Corollary 6.9 of [6] since their forbidden subgraph H is a supergraph of $K_{s,t}^k$.

LEMMA 2.4. *Let G be a $K_{s,t}^k$-free K-almost-regular graph on n vertices with minimum degree $\delta = \omega(1)$. Then provided that L is sufficiently large compared to s, t, k, and K, for any $2 \le \ell \le k$, the number of ℓ-critical paths is at most $n\frac{2(K\delta)^\ell}{f(\ell-1,L)}$.*

Lemma 2.4 roughly says that if a graph has many short critical paths, then we can build a copy of $K_{s,t}^k$. To prove our main theorem, Theorem 1.1, we will need to expand the above concepts introduced by Conlon, Janzer, and Lee as follows.

A *nonpath spider* is a tree with exactly one vertex w of degree at least three, called the *center*. Paths from the center to the leaves are called *legs*. A spider in which all legs have length h is called a *spider of height h*. In this paper, we usually specify the leaves of a spider T in an order as a vector $(v_1, \dots, v_m)$ and call it the *leaf vector* of T. Once the leaf vector is specified, for each i, we call the leg from the center of T to v_i the *ith leg* of T and denote its length by ℓ_i. We then call $(\ell_1, \dots, \ell_m)$ the *length vector* of T. So, a spider becomes leg-labeled once its leaf vector is specified.

DEFINITION 2.5. *Let $s \ge 3$, $k \ge 2$ be integers. Let G be a graph. A spider in G is* feasible *if it contains no critical subpath of length at most k as defined in Definition 2.2. Let $\vec{\ell} := (\ell_1, \dots, \ell_s)$ be a vector of s positive integers, each of which is at most k. We say that a vector of distinct vertices $(v_1, \dots, v_s)$ in G is $(\ell_1, \dots, \ell_s)$-strong if G contains at least $(sk)^{sk-\ell} \cdot f(k, L)$ internally vertex-disjoint feasible spiders with leaf vector $(v_1, \dots, v_s)$ and length vector $(\ell_1, \dots, \ell_s)$, where $\ell = \ell_1 + \cdots + \ell_s$. A spider with leaf vector $(v_1, \dots, v_s)$ and length vector $(\ell_1, \dots, \ell_s)$ is called $(\ell_1, \dots, \ell_s)$-strong if it is feasible and its leaf vector $(v_1, \dots, v_s)$ is $(\ell_1, \dots, \ell_s)$-strong. As the length vector of any spider is fixed, when we say a spider is strong, it is understood that it is strong relative to its length vector.*

DEFINITION 2.6. *Let $s \ge 3$ be an integer. Let $(v_1, \dots, v_s)$ be a vector of s distinct vertices. Let F be a spider with center w and leaf vector $(v_1, \dots, v_s)$ and length vector $(\ell_1, \dots, \ell_s)$. Let $(j_1, \dots, j_s)$ be a vector of integers, where for each $i \in [s], 0 \le j_i \le \ell_i$. We define the $(j_1, \dots, j_s)$-truncation of F, denoted by $F_{(j_1,\dots,j_s)}$, to be a leg-labeled s-legged spider F' with center w obtained by taking its ith leg to be the subpath of length j_i starting at w along the ith leg of F for each $i \in [s]$.*

Note that in Definition 2.6, we allow a leg to have length 0 (and thus be the vertex w itself). In most applications, we consider only truncations in which all legs still have positive lengths. The only exception occurs in one place of the proof of Lemma 3.4, where the allowance of a zero length leg is purely for notational convenience.

Next, we give a few preliminary lemmas.

LEMMA 2.7. *Let G be a K-almost-regular graph with minimum degree δ. Let x be a vertex. Let $\mathcal{C}$ be a family of at least $\alpha\delta^h$ distinct paths of length h with one end x. Then $\mathcal{C}$ contains a subfamily $\mathcal{D}$ of more than $(\alpha/hK^{h-1})\delta$ paths which are vertex-disjoint outside $\{x\}$.*

Proof. Let $\mathcal{D} \subseteq \mathcal{C}$ be a maximal subfamily of paths that are vertex disjoint outside $\{x\}$. Let W be the set of vertices contained in these paths except x. Then $|W| = h|\mathcal{D}|$. By the maximality of $\mathcal{D}$ each member of $\mathcal{C}$ must pass through x and some vertex in W. Since G has maximum degree at most $K\delta$, there can be at most $|W|(K\delta)^{h-1}$ such paths. Hence $|\mathcal{C}| \leq |W|(K\delta)^{h-1}$. Since $|\mathcal{C}| \geq \alpha\delta^h$ and $|W| = h|\mathcal{D}|$, we have $|\mathcal{D}| \geq (\alpha/hK^{h-1})\delta$. ∎

LEMMA 2.8. *Let G be a K-almost-regular graph with minimum degree δ. Let x be a vertex. Let $\mathcal{C}$ be a family of at least $\alpha\delta^h$ distinct paths of length h with one end x and another end in a set S. For each $i \in [h]$ there exists a vertex x_i and a spider of height i with center x_i and leaves in S which has at least $(\alpha/hK^{h-1})\delta$ legs. Furthermore, if $i \neq h$, then $x_i \neq x$.*

Proof. Since G has maximum degree at most $K\delta$, there are at most $(K\delta)^{h-i}$ distinct paths of length $h-i$ starting at x. So there is a path Q of length $h-i$ starting at x and ending at some vertex x_i that is the initial segment of at least $|\mathcal{C}|/(K\delta)^{h-i} \geq (\alpha/K^{h-i})\delta^i$ members of $\mathcal{C}$. If $i \neq h$, then $x_i \neq x$. Let $\mathcal{C}'$ denote the subfamily consisting of these members. Then $\{P-(V(Q)-\{x_i\}) : P \in \mathcal{C}'\}$ is a family of $|\mathcal{C}'|$ distinct paths of length i each of which starts at x_i and ends in S. By Lemma 2.7, $\mathcal{C}'$ contains a subfamily of size at least $[(\alpha/K^{h-i})/iK^{i-1}]\delta \geq (\alpha/hK^{h-1})\delta$ which are vertex-disjoint outside $\{x_i\}$. The claim holds. ∎

An s-uniform hypergraph $\mathcal{F}$ is called s-partite if there exists a partition of $V(\mathcal{F})$ into $A_1, \ldots, A_s$ such that each edge contains one vertex from each A_i. We call the A_i's the parts.

LEMMA 2.9. *Let $\mathcal{F}$ be an s-partite s-graph with parts $A_1, \ldots, A_s$. Suppose that $|\mathcal{F}| > \alpha|A_1|\cdots|A_s|$, where $\alpha > 0$. Let $i \in [s]$. Then there exists a subgraph $\mathcal{F}'$ such that $|\mathcal{F}'| \geq (1/2)|\mathcal{F}|$ and for each $v \in A_i \cap V(\mathcal{F}')$, $d_{\mathcal{F}'}(v) > (\alpha/2)\prod_{j\in[s]\setminus\{i\}}|A_j|$.*

Proof. Let us call a vertex $v \in A_i$ bad if its degree in the remaining graph is at most $(\alpha/2)\prod_{j\neq i}|A_j|$. As long as there exists a bad vertex, we delete this vertex from A_i. Let $\mathcal{F}'$ be the remaining subgraph. Then at most $(\alpha/2)\prod_{j=1}^{s}|A_j|$ edges are removed in the process. So $|\mathcal{F}'| > (1/2)|\mathcal{F}|$. Clearly $\mathcal{F}'$ satisfies the degree requirement. ∎

Note that one could easily modify Lemma 2.9 to apply to all parts. But it suffices for our purposes.

3. Proof of Theorem 1.1.

We break the proof of Theorem 1.1 into two parts.

3.1. Building subdivisions using strong spiders, the general case.

LEMMA 3.1. *Let $K \geq 1, k, t \geq 2, s \geq 3$ be fixed integers. Then provided that L is sufficiently large compared to $s, t, k,$ and K, for any $\beta > 0$ there exists δ_0 such that the following holds. Suppose that G is a $K^k_{s,t}$-free K-almost-regular graph n vertices with minimum degree $\delta \geq \delta_0$. If $\ell_1, \ldots, \ell_s$ are positive integers satisfying that $\forall i \in [s], k/2 \leq \ell_i \leq k$ and that $\forall 1 \leq i < j \leq s, \ell_i + \ell_j \geq k+1$, then the number of tuples $(w, v_1, \ldots, v_s)$ such that there is an $(\ell_1, \ldots, \ell_s)$-strong spider with center w and leaf vector $(v_1, \ldots, v_s)$ is at most $\beta n\delta^\ell$, where $\ell = \ell_1 + \cdots + \ell_s$.*

Proof. For each vertex w in G, let $\mathcal{H}_w$ denote the family of tuples $(v_1, \ldots, v_s)$ such that there is an $(\ell_1, \ldots, \ell_s)$-strong spider with center w and leaf vector $(v_1, \ldots, v_s)$. Suppose for contradiction that there exist more than $\beta n\delta^\ell$ tuples $(w, v_1, \ldots, v_s)$ such that there is an $(\ell_1, \ldots, \ell_s)$-strong spider with center w and leaf vector $(v_1, \ldots, v_s)$. Then by the pigeonhole principle, there exists a vertex w such that $|\mathcal{H}_w| > \beta\delta^\ell$. Let us fix such a w. For each $(v_1,\dots,v_s)\in\mathcal{H}_w$, by definition, we may fix an $(\ell_1,\dots,\ell_s)$-strong spider $T(v_1,\dots,v_s)$ with leaf vector $(v_1,\dots,v_s)$. For each i, we call the path in $T(v_1,\dots,v_s)$ from w to v_i its ith leg.

Randomly and independently color vertices of G with colors $1,\dots,s$ with each vertex receiving each color with probability $1/s$. For each s-tuple $(v_1,\dots,v_s)\in\mathcal{H}_w$, we call it *good* if for each $i\in[s]$ all the vertices on the ith leg of $T(v_1,\dots,v_s)$ except w are colored i. Since $T(v_1,\dots,v_s)-\{w\}$ has ℓ vertices, the probability of $(v_1,\dots,v_s)$ being good is $1/s^\ell$. Hence, there exists a coloring c such that the family

$$\mathcal{F}_w=\{(v_1,\dots,v_s)\in\mathcal{H}_w:(v_1,\dots,v_s)\text{ is good}\}$$

satisfies

$$|\mathcal{F}_w|\ge|\mathcal{H}_w|/s^\ell>(\beta/s^\ell)\delta^\ell. \tag{3.1}$$

Let us fix such a coloring c. For each $i\in[s]$, let

$$A_i=\{v\in V(\mathcal{F}_w):c(v)=i\}.$$

Then $\mathcal{F}_w$ is an s-partite s-graph with parts $A_1,\dots,A_s$. By our assumption, for each $i\in[s]$ and each $v\in A_i$ there is an $(\ell_1,\dots,\ell_s)$-strong spider with center w where v plays the role of the ith vertex in the leaf vector. Furthermore, all the vertices on the ith leg, except w, are colored i under c. Since G has maximum degree at most $K\delta$, we have

$$\forall i\in[s],\ |A_i|\le(K\delta)^{\ell_i}. \tag{3.2}$$

Let $\alpha=\frac{\beta}{s^\ell K^\ell}$. For each $i\in[s]$, let $\alpha_i=\frac{\beta}{s^\ell K^{\ell-\ell_i}}$. By (3.1) and (3.2), we have

$$|\mathcal{F}_w|>\alpha|A_1|\cdots|A_s|\quad\text{ and }\quad\forall i\in[s],|A_i|\ge|\mathcal{F}_w|/\prod_{j\ne i}|A_j|\ge\alpha_i\delta^{\ell_i}. \tag{3.3}$$

Now, we may assume without loss of generality that $\ell_1\le\ell_2\le\cdots\le\ell_s$. First, let us observe that if $\ell_1=\ell_2=\cdots=\ell_s=k$, then we may take any $(k,\dots,k)$-strong tuple $(v_1,\dots,v_s)$. By the definition of strong tuples, there are at least $f(k,L)$ internally vertex-disjoint spiders with leaf vector $(v_1,\dots,v_s)$. It is easy to see that the union of any t of these spiders forms a copy of $K^k_{s,t}$, contradicting G being $K^k_{s,t}$-free. Hence, we may assume that $\ell_1<k$. For each $i\in[s]$, let $m_i=k-\ell_i$. By our assumption, $\forall i\in[s],\ell_i\ge k/2$, and $\forall i,j\in[s],\ell_i+\ell_j>k$. This implies that

$$m_1\le\ell_1\text{ and }\forall 2\le i\le s, m_i<\ell_i.$$

Let $q=\max\{i:\ell_i<k\}$. Then $1\le q\le s$. By Lemma 2.9, $\mathcal{F}_w$ contains a subgraph $\mathcal{F}_1$ such that

$$|\mathcal{F}_1|>(1/2)|\mathcal{F}_w|>(\alpha/2)|A_1|\cdots|A_s| \tag{3.4}$$

and

$$\forall v\in A_1':=A_1\cap V(\mathcal{F}_1),\ d_{\mathcal{F}_1}(v)\ge(\alpha/2)\prod_{j\ne1}|A_j|. \tag{3.5}$$

By (3.4) and (3.3), we have

$$|A_1'|\ge(\alpha/2)|A_1|\ge(1/2)\alpha\alpha_1\delta^{\ell_1}. \tag{3.6}$$

For each $v \in A_1'$ there is an edge of $\mathcal{F}_1$ containing it, which in particular, by our earlier discussion, implies that there is a path P_v of length ℓ_1 from w to v, all of whose vertices except w are colored i by c. Let

$$\beta_1 = \frac{(1/2)\alpha\alpha_1}{\ell_1 K^{\ell_1 - 1}}.$$

By Lemma 2.8, there exist a vertex z_1 and a spider S_1 of height m_1 with center at z_1 and leaf set $B_1 \subseteq A_1'$ such that

$$\beta_1\delta \le |B_1| \le \delta.$$

Note that if $m_1 = \ell_1$, then $z_1 = w$. If $m_1 < \ell_1$, then $z_1 \neq w$. Also, all the vertices in S_1, except possibly w, have color 1 in c. Let $\mathcal{F}_1'$ be the subgraph of $\mathcal{F}_1$ induced by the parts $B_1, A_2, \ldots, A_s$. Since $B_1 \subseteq A_1'$, by (3.5) we have that

$$\forall v \in B_1,\ d_{\mathcal{F}_1'}(v) \ge (\alpha/2) \prod_{j \neq 1} |A_j|. \tag{3.7}$$

In general, let $1 \le i \le q-1$ and suppose we have defined $\mathcal{F}_1', \ldots, \mathcal{F}_i'$ and $B_1, \ldots, B_i$, where each $\mathcal{F}_j'$ has parts $B_1, \ldots, B_j, A_{j+1}, \ldots, A_s$ and satisfies that

$$\forall v \in B_j,\ d_{\mathcal{F}_j'}(v) \ge (\alpha/2^i)|B_1| \cdots |B_{j-1}||A_{j+1}| \cdots |A_s| \tag{3.8}$$

and

$$\beta_j\delta \le |B_j| \le \delta, \text{ where } \beta_j = \frac{(1/2^j)\alpha\alpha_j}{\ell_j K^{\ell_j - 1}}.$$

Also, suppose that there are distinct vertices $z_1, \ldots, z_i$ such that for each $j \in [i]$, there is a spider S_j of height m_j with center z_j and leaf set B_j, all of whose vertices except possibly w lie in color class j of c. Also, suppose that $z_2, \ldots, z_i \neq w$ and $z_1 = w$ if and only if $\ell_1 = m_1$. By (3.8),

$$|\mathcal{F}_i'| \ge (\alpha/2^i)|B_1| \cdots |B_i||A_{i+1}| \cdots |A_s|.$$

By Lemma 2.9, $\mathcal{F}_i'$ contains a subgraph $\mathcal{F}_{i+1}$ such that

$$|\mathcal{F}_{i+1}| \ge (1/2)|\mathcal{F}_i'| \ge (\alpha/2^{i+1})|B_1| \cdots |B_i||A_{i+1}| \cdots |A_s| \tag{3.9}$$

and

$$\forall v \in A_{i+1}' := A_{i+1} \cap V(\mathcal{F}_{i+1}),\ d_{\mathcal{F}_{i+1}}(v) \ge (\alpha/2^{i+1})|B_1| \cdots |B_i||A_{i+2}| \cdots |A_s|. \tag{3.10}$$

By (3.10) and (3.3) we have

$$|A_{i+1}'| \ge (\alpha/2^{i+1})|A_{i+1}| \ge (1/2^{i+1})\alpha\alpha_{i+1}\delta^{\ell_{i+1}}. \tag{3.11}$$

As before, for each $v \in A_{i+1}'$ there is a path P_v of length ℓ_{i+1} from w to v, all of whose vertices except w have color $i+1$ in c. Let

$$\beta_{i+1} = \frac{(1/2^{i+1})\alpha\alpha_{i+1}}{\ell_{i+1} K^{\ell_{i+1} - 1}}.$$

By Lemma 2.8, there exist a vertex z_{i+1} and a spider S_{i+1} of height m_{i+1} with center z_{i+1} and leaf set $B_{i+1} \subseteq A_{i+1}'$ such that

$$\beta_{i+1}\delta \le |B_{i+1}| \le \delta.$$

Furthermore, since $m_{i+1} < \ell_{i+1}$, we have $z_{i+1} \neq w$. Also, all the vertices in S_{i+1} lie in color class $i+1$ of c. Since $B_{i+1} \subseteq A'_{i+1}$, by (3.10)

$$\forall v \in B_{i+1},\ d_{\mathcal{F}_{i+1}}(v) \geq (\alpha/2^{i+1})|B_1|\cdots|B_i||A_{i+2}|\cdots|A_s|.$$

Finally, let $\mathcal{F}'_{i+1}$ be the subgraph of $\mathcal{F}_{i+1}$ induced by the parts $B_1, \ldots, B_{i+1}, A_{i+2}, \ldots, A_s$. This allows to define $\mathcal{F}'_1, \ldots, \mathcal{F}'_q$, $B_1, \ldots, B_q$, and $z_1, \ldots, z_q$. Now, we claim that we can find a copy of $K^k_{s,t}$ in G, which would give us a contradiction. To find such a copy, we consider two cases.

Case 1. $q = s$.

By our assumption, $\mathcal{F}'_s$ is an s-partite s-graph with parts $B_1, \ldots, B_s$, where

$$|\mathcal{F}'_s| \geq (\alpha/2^s)|B_1|\cdots|B_s|$$

and

$$\forall i \in [s],\ \beta_i\delta \leq |B_i| \leq \delta, \text{ where } \beta_i = \frac{(1/2^i)\alpha\alpha_i}{\ell_i K^{\ell_i - 1}}.$$

Let $\mathcal{M}$ be a maximum matching in $\mathcal{F}'_s$. Then the maximality of $\mathcal{M}$ implies that every edge of $\mathcal{F}'_s$ contains some vertex in $V(\mathcal{M})$. On the other hand, since $\mathcal{F}'_s$ is s-partite and each part has size at most δ, each vertex is contained in at most δ^{s-1} edges. Hence

$$|\mathcal{F}'_s| \leq |V(\mathcal{M})| \cdot \delta^{s-1} = s|\mathcal{M}|\delta^{s-1}.$$

Hence by the above lower bounds on $|\mathcal{F}'_s|$ and $|B_1|, \ldots, |B_s|$, we have

$$|\mathcal{M}| \geq |\mathcal{F}'_s|/(s\delta^{s-1}) \geq (2^{-s}\alpha\beta_1\cdots\beta_s/s)\delta \gg t$$

for sufficiently large δ (as $\delta \geq \delta_0$). Let $\mathcal{M}'$ be a set of t edges in $\mathcal{M}$. Suppose $\mathcal{M}' = \{e_1, \ldots, e_t\}$. For each $i \in [t]$, suppose $e_i = (v^i_1, v^i_2, \ldots, v^i_s)$, where $\forall j \in [s], v^i_j \in B_j$. For each $j \in [s]$, let Z_j be the subspider of S_j obtained by keeping only the t paths from z_j to $V(\mathcal{M}') \cap B_j$. Since vertices in $Z_1 - \{w\}$ have color 1 and for each $2 \leq j \leq s$, vertices in Z_j have color j, $Z_1, \ldots, Z_t$ are vertex-disjoint.

By the definition of $\mathcal{F}'_s \subseteq H_w$, for each $i \in [t]$, $(v^i_1, \ldots, v^i_s)$ is an $(\ell_1, \ldots, \ell_s)$-strong tuple and hence there are $f(k, L)$ internally vertex-disjoint spiders with leaf vector $(v^i_1, \ldots, v^i_s)$ and length vector $(\ell_1, \ldots, \ell_s)$. Since $f(k, L) \gg |V(K^k_{s,t})|$, we can greedily find t vertex disjoint spiders $T_1, \ldots, T_t$ such that for each $i \in [t]$, T_i has leaf vector $(v^i_1, \ldots, v^i_s)$ and length vector $(\ell_1, \ldots, \ell_s)$ and that $V(T_i) \setminus \{v^i_1, v^i_2, \ldots, v^i_s\}$ is disjoint from $\bigcup_{j=1}^s V(Z_j)$. Now $(\bigcup_{i=1}^t T_i) \cup (\bigcup_{j=1}^s Z_j)$ forms a copy of $K^k_{s,t}$, contradicting G being $K^k_{s,t}$-free.

Case 2. $q < s$.

Since $|\mathcal{F}'_q| \geq (\alpha/2^q)|B_1|\cdots|B_q||A_{q+1}|\cdots|A_s|$, by averaging, there exists a tuple $(z_{q+1}, \ldots, z_s) \in A_{q+1} \times \cdots \times A_s$ that is contained in at least $(\alpha/2^s)|B_1|\cdots|B_q|$ of the edges of $\mathcal{F}'_q$. Let

$$\mathcal{F}^* = \{e \setminus \{z_{q+1}, \ldots, z_s\} : \{z_{q+1}, \ldots, z_s\} \subseteq e \in \mathcal{F}'_q\}.$$

As in Case 1, for sufficiently large δ, $\mathcal{F}^*$ contains a matching $\mathcal{M} = \{e_1, \ldots, e_t\}$ of size t.

For each $i \in [t]$, suppose $e_i = (v^i_1, v^i_2, \ldots, v^i_q)$, where $\forall j \in [q], v^i_j \in B_j$. For each $j \in [q]$, let Z_j be the subspider of S_j obtained by keeping only the t paths from z_j to $V(\mathcal{M}) \cap B_j$. Since vertices in $Z_1 - \{w\}$ have color 1 and for each $2 \leq j \leq s$, vertices in Z_j have color j, $Z_1, \ldots, Z_q$ are vertex-disjoint.

By definition, for each $i \in [t]$, $(v_1^i, \ldots, v_q^i, z_{q+1}, \ldots, z_s)$ is an $(\ell_1, \ldots, \ell_s)$-strong tuple and hence there are $f(k, L)$ internally vertex-disjoint spiders with leaf vector $(v_1^i, \ldots, v_q^i, z_{q+1}, \ldots, z_s)$ and length vector $(\ell_1, \ldots, \ell_s)$. Since $f(k, L) \gg |V(K_{s,t}^k)|$, we can greedily find t spiders $T_1, \ldots, T_t$ such that for each $i \in [t]$, T_i has leaf vector $(v_1^i, \ldots, v_q^i, z_{q+1}, \ldots, z_s)$ and length vector $(\ell_1, \ldots, \ell_s)$ and that $V(T_i) \setminus \{z_{q+1}, \ldots, z_s\}$ are pairwise disjoint over different i and that $V(T_i) \setminus \{v_1^i, \ldots, v_q^i, z_{q+1}, \ldots, z_s\}$ is disjoint from $\bigcup_{j=1}^q V(Z_j)$ for each $i \in [t]$. Now $(\bigcup_{i=1}^t T_i) \cup (\bigcup_{j=1}^q Z_j)$ forms a copy of $K_{s,t}^k$, contradicting G being $K_{s,t}^k$-free. ∎

From Lemma 3.1, we immediately obtain the following.

COROLLARY 3.2. *Let $K \geq 1$ and integers $k, t \geq 2, s \geq 3$ be fixed. Then provided that L is sufficiently large compared to s, t, k, and K, for any $\beta > 0$ there exists δ_0 such that the following holds. Suppose that G is a $K_{s,t}^k$-free K-almost-regular graph on n vertices with minimum degree $\delta \geq \delta_0$. Suppose $\ell_1, \ldots, \ell_s$ are positive integers satisfying that $\forall i \in [s], k/2 \leq \ell_i \leq k$ and that $\forall 1 \leq i < j \leq s, \ell_i + \ell_j \geq k+1$. Let $\mathcal{F}$ denote the family of s-legged leg-labeled feasible spiders F of height k that satisfy that $F_{(\ell_1,\ldots,\ell_s)}$ is $(\ell_1, \ldots, \ell_s)$-strong. Then $|\mathcal{F}| \leq [K^k f(k, L)]^s \beta n \delta^{ks}$.*

Proof. Let $\mathcal{S}$ be the family of $(\ell_1, \ldots, \ell_s)$-strong spiders in G. Let $\ell = \ell_1 + \cdots + \ell_s$. By Lemma 3.1, there are at most $\beta n \delta^\ell$ tuples $(w, v_1, \ldots, v_s)$ such that there is a member of $\mathcal{S}$ that has w as the center and $(v_1, \ldots, v_s)$ as the leaf vector. For each i, there are at most $f(\ell_i, L) \leq f(k, L)$ light paths of length ℓ_i in G between w and v_i. Since the members of $\mathcal{S}$ are strong and in particular are feasible, every leg of members of $\mathcal{S}$ is light. It follows that $|\mathcal{S}| \leq [f(k, L)]^s \beta n \delta^\ell$.

Now, let $F \in \mathcal{F}$. By definition of $\mathcal{F}$, $F_{(\ell_1,\ldots,\ell_s)}$ is a member of $\mathcal{S}$. Since G has maximum degree at most $K\delta$, for each $F \in \mathcal{F}$, there are at most $(K\delta)^{ks-\ell}$ members $F' \in \mathcal{F}$ with $F_{(\ell_1,\ldots,\ell_s)} = F'_{(\ell_1,\ldots,\ell_s)}$. Hence $|\mathcal{F}| \leq (K\delta)^{ks-\ell} |\mathcal{S}| \leq (K\delta)^{ks-\ell} [f(k, L)]^s \beta n \delta^\ell \leq [K^k f(k, L)]^s \beta n \delta^{ks}$. ∎

3.2. Building subdivisions using strong spiders: The $(1, k, \ldots, k)$-case.

In this section, we prove a second crucial ingredient (Lemma 3.4 below) which complements Lemma 3.1. First we need an auxiliary lemma.

LEMMA 3.3. *Let $\mathcal{F}$ be a family of feasible spiders that have the same leaf vector $(v_1, \ldots, v_s)$ and length vector $(\ell_1, \ldots, \ell_s)$, where each $1 \leq \ell_i \leq k$. Let $\ell = \ell_1 + \cdots + \ell_s$. If $|\mathcal{F}| \geq [(sk)^{sk} \cdot f(k, L)^2]^\ell$, then there exist a vector of positive integers $(j_1, \ldots, j_s)$ and a vector of distinct vertices $(y_1, \ldots, y_s)$ such that the family*

$$\{F_{(j_1,\ldots,j_s)} : F \in \mathcal{F} \text{ and } F_{(j_1,\ldots,j_s)} \text{ has leaf vector } (y_1, \ldots, y_s)\}$$

contains at least $(sk)^{sk-j} f(k, L)$ internally disjoint members, where $j = j_1 + \cdots + j_s$.

Proof. We prove it by induction on ℓ. The case of $\ell = s$ is trivial. Assume $\ell > s$ and assume that claim holds for smaller ℓ values. Now pick a maximal family $\mathcal{M}$ of internally disjoint spiders in $\mathcal{F}$. If $|\mathcal{M}| \geq (sk)^{sk-\ell} \cdot f(k, L)$, then let $(y_1, \ldots, y_s) = (v_1, \ldots, v_s)$ and $(j_1, \ldots, j_s) = (\ell_1, \ldots, \ell_s)$ and we are done. So we may assume $|\mathcal{M}| < (sk)^{sk-\ell} \cdot f(k, L)$. Let U be the set of internal vertices of the spiders in $\mathcal{M}$. Then $|U| \leq sk \cdot |\mathcal{M}| < (sk)^{sk-\ell+1} \cdot f(k, L)$. By the maximality of $\mathcal{M}$, any spider in $\mathcal{F}$ contains a vertex in U as an internal vertex. So by averaging, there exists $u \in U$ such that the size of the family $\mathcal{F}'$ which consists of all spiders in $\mathcal{F}$ that contain u as an internal vertex is at least

$$|\mathcal{F}'| \geq \frac{|\mathcal{F}|}{|U|} \geq \frac{|\mathcal{F}|}{(sk)^{sk-\ell+1} \cdot f(k, L)}.$$

By averaging again, there is a subfamily $\mathcal{F}'' \subseteq \mathcal{F}'$ of size

$$|\mathcal{F}''| \geq \frac{|\mathcal{F}'|}{\ell - s + 1} \geq \frac{|\mathcal{F}|}{(sk)^{sk-\ell+2} \cdot f(k, L)}$$

such that u plays the same role in all the members of $\mathcal{F}''$. Since $\mathcal{F}'' \subseteq \mathcal{F}$ and the members of $\mathcal{F}$ are all feasible, every leg of members of $\mathcal{F}''$ is light. Hence there are no more than $\prod_{i=1}^{s} f(\ell_i, L) \leq [f(k, L)]^s$ members of $\mathcal{F}''$ that contain u as their center. It is easy to check by our assumption on $|\mathcal{F}|$ that $|\mathcal{F}''| > [f(k, L)]^s$. So u cannot be the center of the spiders in $\mathcal{F}''$. Without loss of generality, suppose that u is in the first leg of each member of $\mathcal{F}''$ and that the distance between u and the center of the member is $\ell_1' < \ell_1$. Let

$$\mathcal{J} = \{F_{(\ell_1', \ell_2, \ldots, \ell_s)} : F \in \mathcal{F}''\}.$$

By the definition of $\mathcal{F}''$ and $\mathcal{J}$, all members of $\mathcal{J}$ have the leaf vector $(u, v_2, \ldots, v_s)$. Since members of $\mathcal{F}''$ are feasible, each member of $\mathcal{F}''$ is the union of a member of $\mathcal{J}$ and a $(\ell_1 - \ell_1')$-light path between u and v_1. Hence each member of $\mathcal{J}$ is contained in fewer than $f(\ell_1 - \ell_1', L) \leq f(k, L)$ members of $\mathcal{F}''$. Therefore,

$$|\mathcal{J}| \geq \frac{|\mathcal{F}''|}{f(k, L)} \geq \frac{|\mathcal{F}|}{(sk)^{sk} \cdot f(k, L)^2} \geq [(sk)^{sk} \cdot f(k, L)^2]^{\ell - 1}.$$

Since $j := \ell_1' + \ell_2 + \cdots + \ell_s \leq \ell - 1$ and $|\mathcal{J}| \geq [(sk)^{sk} \cdot f(k, L)^2]^j$, by the induction hypothesis, $\mathcal{J}$ provides at least $(sk)^{sk-j} \cdot f(k, L)$ internally disjoint subspiders with the same length vector and leaf vector. This completes the proof. ∎

LEMMA 3.4. *Let $K \geq 1$ and integers $k, s, t \geq 2$ be fixed. Then provided that L is sufficiently large compared to s, t, k, and K, for any $\gamma > 0$ there exist $n_0, C > 0$ such that the following holds. Let $\mathcal{A} = \{(j_1, \ldots, j_s) : \forall i \in [s], j_i \in [k]\}$. Let $\mathcal{L}$ be the set of all s-tuples consisting of one 1 and $s - 1$ many k's. Let G be a $K_{s,t}^k$-free K-almost-regular graph on $n \geq n_0$ vertices with minimum degree $\delta \geq Cn^{\frac{1}{k} - \frac{1}{sk}}$. Let $\mathcal{F}$ be the family of all the s-legged leg-labeled feasible spiders F of height k in G that satisfy the following:*

1. *For some $(j_1, \ldots, j_s) \in \mathcal{L}$, $F_{(j_1, \ldots, j_s)}$ is $(j_1, \ldots, j_s)$-strong.*
2. *For all $(j_1, \ldots, j_s) \in \mathcal{A} \setminus \mathcal{L}$, $F_{(j_1, \ldots, j_s)}$ is not $(j_1, \ldots, j_s)$-strong.*

Then $|\mathcal{F}| \leq k\gamma n\delta^{sk}$.

Proof. For each $i \in [s]$, let $\mathcal{F}_i$ denote the subfamily of members F of $\mathcal{F}$ such that $F_{(j_1, \ldots, j_s)}$ is $(j_1, \ldots, j_s)$-strong when $j_i = 1$ and $\forall \ell \in [k] \setminus \{i\}, j_\ell = k$. Then $\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{F}_i$. Suppose to the contrary that $|\mathcal{F}| \geq k\gamma n\delta^{ks}$. By averaging, there exists some $i \in [k]$ such that $|\mathcal{F}_i| \geq \frac{|\mathcal{F}|}{k} \geq \gamma n\delta^{sk}$. Without loss of generality, we may assume that $|\mathcal{F}_1| \geq \gamma n\delta^{sk}$. We derive a contradiction.

Let $c > 0$ such that $2cK^{sk} = \frac{\gamma}{4}$. We do some cleaning to $\mathcal{F}_1$ through member removals. We update $\mathcal{F}_1$ immediately after each removal. At any point, if there is a member $F \in \mathcal{F}_1$ such that $F_{(1,k,\ldots,k)} = F'_{(1,k,\ldots,k)}$ for fewer than $c\delta(K\delta)^{k-2}$ members F' of $\mathcal{F}_1$ we remove all these members F from $\mathcal{F}_1$. Whenever there is a member $F \in \mathcal{F}_1$ such that $F_{(0,k,\ldots,k)} = F'_{(0,k,\ldots,k)}$ for fewer than $c\delta(K\delta)^{k-1}$ members $F' \in \mathcal{F}_1$, we remove all these members F' from $\mathcal{F}_1$. We continue the process until no further removal can be performed. Let $\mathcal{F}'$ denote the final $\mathcal{F}_1$.

Since G has maximum degree at most $K\delta$, the total number of members we have removed from $\mathcal{F}_1$ is no more than $n(K\delta)^{(s-1)k+1} \cdot c\delta(K\delta)^{k-2} + n(K\delta)^{(s-1)k} \cdot c\delta \cdot$

$(K\delta)^{k-1} \le 2cK^{ks}n\delta^{ks} = \frac{1}{4}\gamma n\delta^{ks}$. Hence

$$|\mathcal{F}'| \ge \gamma n\delta^{ks} - \frac{1}{4}\gamma n\delta^{ks} = \frac{3}{4}\gamma n\delta^{ks}.$$

Given an $(s-1)$-tuple $\vec{a} = (a_2, \ldots, a_s)$ of vertices in G, let

$$\mathcal{F}_{\vec{a}} = \{F \in \mathcal{F}' : F_{(1,k,\ldots,k)} \text{ has leaf vector } (u, a_2, \ldots, a_s) \text{ for some vertex } u\}.$$

Let $F \in \mathcal{F}_{\vec{a}}$. By definition, $F_{(1,k,\ldots,k)}$ is $(1,k,\ldots,k)$-strong and has leaf vector $(u, a_2, \ldots, a_s)$ for some vertex u. Let $w(F)$ denote the center of F and $u(F)$ denote the vertex u. Let

$$H_{\vec{a}} = \{w(F)u(F) : F \in \mathcal{F}_{\vec{a}}\}.$$

Furthermore, let

$$W_{\vec{a}} = \{w(F) : F \in \mathcal{F}_{\vec{a}}\} \quad \text{and} \quad U_{\vec{a}} = \{u(F) : F \in \mathcal{F}_{\vec{a}}\}.$$

Hence, $H_{\vec{a}}$ is a graph on $W_{\vec{a}} \cup U_{\vec{a}}$. By definition, for each $u \in U_{\vec{a}}$ there exists some $F \in \mathcal{F}'$ such that $F_{(1,k,\ldots,k)}$ is $(1,k,\ldots,k)$-strong and has leaf vector $(u, a_2, \ldots, a_s)$. So, in particular,

$$\text{for any } u \in U_{\vec{a}},\ (u, a_2, \ldots, a_s) \text{ is } (1,k,\ldots,k)\text{-strong.} \tag{3.12}$$

CLAIM 1. *Let $\vec{a}$ be an $(s-1)$-tuple such that $\mathcal{F}_{\vec{a}} \ne \emptyset$. Let $uw \in H_{\vec{a}}$, where $u \in U_{\vec{a}}$ and $w \in W_{\vec{a}}$. Then the number of $F \in \mathcal{F}_{\vec{a}}$ with $(u(F), w(F)) = (u, w)$ is at least $c\delta(K\delta)^{k-2}$ and at most $[f(k,L)]^{s-1} \cdot (K\delta)^{k-1}$.*

Proof of Claim 1. By definition, there is a member $F \in \mathcal{F}_{\vec{a}}$ such that $w(F) = w$ and $u(F) = u$. By our cleaning rule in forming $\mathcal{F}'$, there are at least $c\delta(K\delta)^{k-2}$ members F' of $\mathcal{F}$ with $F_{(1,k,\ldots,k)} = F'_{(1,k,\ldots,k)}$. For each such F', clearly, $F' \in \mathcal{F}_{\vec{a}}$ and $(u(F'), w(F')) = (u, w)$. So, the first part of the claim holds.

We now prove the second part of the claim. Since members of $\mathcal{F}_{\vec{a}}$ are feasible spiders, to form a member $F \in \mathcal{F}_{\vec{a}}$ with $(u(F), w(F)) = (u, w)$, we need to pick a k-light path from w to each of $a_2, \ldots, a_s$ and a $(k-1)$-path starting at u. So, there are at most $[f(k,L)]^{s-1} \cdot (K\delta)^{k-1}$ such F. ∎

CLAIM 2. *For each $(s-1)$-tuple $\vec{a} = (a_2, \ldots, a_s)$ with $\mathcal{F}_{\vec{a}} \ne \emptyset$, we have that $|\mathcal{F}_{\vec{a}}| \ge e(H_{\vec{a}}) \cdot c\delta(K\delta)^{k-2}$ and $e(H_{\vec{a}}) \ge c\delta \cdot |W_{\vec{a}}|$.*

Proof of Claim 2. By Claim 1, for each $wu \in H_{\vec{a}}$ there are at least $c\delta(K\delta)^{k-2}$ members F of $\mathcal{F}_{\vec{a}}$ with $(u(F), w(F)) = (u, w)$. As different wu's clearly give rise to different F's, the first part of the claim follows.

Now, let $w \in W_{\vec{a}}$. By definition of $\mathcal{F}_{\vec{a}}$, there is a member $F \in \mathcal{F}_{\vec{a}}$ such that $w(F) = w$. By our cleaning rule in forming $\mathcal{F}'$, there are at least $c\delta(K\delta)^{k-1}$ members $F' \in \mathcal{F}'$ such that $F_{(0,k,\ldots,k)} = F'_{(0,k,\ldots,k)}$. Clearly, for each such F' we have $F' \in \mathcal{F}_{\vec{a}}$ and $w(F') = w$. Each such F' is the union of $F_{(0,k,\ldots,k)}$ and a path of length k from w with the first edge being $wu(F')$. If there are fewer than $c\delta$ different $u(F')$, then since G has maximum degree at most $K\delta$ the total number of such F' would be fewer than $c\delta(K\delta)^{k-1}$, contradicting our definition of $\mathcal{F}'$. So over all $F' \in \mathcal{F}_{\vec{a}}$ with $F'_{(0,k,\ldots,k)} = F_{(0,k,\ldots,k)}$ there must be at least $c\delta$ different $u(F')$. So w has degree at least $c\delta$ in $H_{\vec{a}}$. This proves the second part of the claim. ∎

For any $(s-1)$-tuple $\vec{a} = (a_2, \ldots, a_s)$, let

$$U^+_{\vec{a}} = \{u \in U_{\vec{a}} : d_{H_{\vec{a}}}(u) \ge 2kt\} \quad \text{and} \quad U^-_{\vec{a}} = \{u \in U_{\vec{a}} : d_{H_{\vec{a}}}(u) < 2kt\}.$$

Let

$$\mathcal{F}_{\vec{a}}^+ = \{F \in \mathcal{F}_{\vec{a}} : u(F) \in U_{\vec{a}}^+\} \quad \text{and} \quad \mathcal{F}_{\vec{a}}^- = \{F \in \mathcal{F}_{\vec{a}} : u(F) \in U_{\vec{a}}^-\}.$$

CLAIM 3. *For every $(s-1)$-tuple $\vec{a}$, let $H_{\vec{a}}^+ = \{w(F)u(F) : F \in \mathcal{F}_{\vec{a}}^+\}$. We have $e(H_{\vec{a}}^+) \leq 2kt|W_{\vec{a}}|$.*

Proof of Claim 3. Let $\vec{a}$ be given. For convenience, let $U^+ = U_{\vec{a}}^+$ and $W = W_{\vec{a}}$. Suppose that $e(H_{\vec{a}}^+) > 2kt|W|$. Then this, together with the definition of $U_{\vec{a}}^+$, implies that the average degree of $H_{\vec{a}}^+$ is at least $2kt$. By a well-known fact, $H_{\vec{a}}^+$ contains a subgraph H' with minimum degree at least kt. Even though W and U^+ may not be disjoint, using $\delta(H') \geq kt$ and the fact that each edge of it has the form wu, where $w \in W$ and $u \in U^+$, by a greedy process, in H', we can build a t-legged spider T of height $k-1$ with leaves lying in U^+. Let x be its center and $u_1, \ldots, u_t$ be its leaves. By (3.12), $(u_i, a_2, \ldots, a_s)$ is $(1, k, \ldots, k)$-strong for every $i \in [t]$. Thus using strongness one can greedily find t internally disjoint spiders of height k with leaf vector $(x, a_2, \ldots, a_s)$. The union of these t spiders forms a copy of $K_{s,t}^k$, contradicting G being $K_{s,t}^k$-free. ∎

By Claims 1 and 3, we have

$$(3.13) \quad |\mathcal{F}_{\vec{a}}^+| \leq e(H_{\vec{a}}^+) \cdot [f(k,L)]^{s-1}(K\delta)^{k-1} \leq [2kt[f(k,L)]^{s-1}K^{k-1}] \cdot |W_{\vec{a}}| \cdot \delta^{k-1}.$$

On the other hand, by Claim 2 we have that

$$|\mathcal{F}_{\vec{a}}| \geq e(H_{\vec{a}}) \cdot c\delta(K\delta)^{k-2} \geq c\delta|W_{\vec{a}}| \cdot c\delta(K\delta)^{k-2} = c^2K^{k-2} \cdot |W_{\vec{a}}| \cdot \delta^k.$$

As $\delta \geq Cn^{\frac{1}{k} - \frac{1}{sk}}$ and $n \geq n_0$ is sufficiently large, this together with (3.13) yields that

$$|\mathcal{F}_{\vec{a}}^+| \leq \frac{1}{2}|\mathcal{F}_{\vec{a}}|.$$

Thus $|\mathcal{F}_{\vec{a}}^-| = |\mathcal{F}_{\vec{a}}| - |\mathcal{F}_{\vec{a}}^+| \geq \frac{1}{2}|\mathcal{F}_{\vec{a}}|$. Since $\mathcal{F}' = \cup_{\vec{a}} \mathcal{F}_{\vec{a}}$, we have that $\sum_{\vec{a}} |\mathcal{F}_{\vec{a}}| \geq |\mathcal{F}'| \geq \frac{3}{4}\gamma n\delta^{sk}$. It follows that

$$\sum_{\vec{a}} |\mathcal{F}_{\vec{a}}^-| \geq \frac{1}{2}\sum_{\vec{a}} |\mathcal{F}_{\vec{a}}| \geq \frac{3}{8}\gamma n\delta^{ks} \geq \frac{3\gamma C^{sk}}{8}n^s.$$

By averaging, there exists an $(s-1)$-tuple $\vec{a}$ such that $|\mathcal{F}_{\vec{a}}^-| \geq C_1 n$ for some constant C_1 that can be made arbitrarily large by taking C to be sufficiently large. By averaging again, there exists some z such that the number of spiders in $\mathcal{F}_{\vec{a}}^-$ with leaf vector $(z, \vec{a})$ is at least C_1. Fix such a vertex z and let

$$\mathcal{F}_{z,\vec{a}} = \{F \in \mathcal{F}_{\vec{a}}^- : F \text{ has leaf vector } (z, \vec{a})\}.$$

By choosing C to be sufficiently large (which makes C_1 arbitrarily large), we can ensure

$$|\mathcal{F}_{z,\vec{a}}| \geq C_1 \geq [(sk)^{sk} \cdot f(k,L)^2]^{sk}.$$

CLAIM 4. *There exists a member F of $\mathcal{F}_{z,\vec{a}}$ and a tuple $(j_1, \ldots, j_s) \in \mathcal{A} \setminus \mathcal{L}$ such that $F_{(j_1,\ldots,j_s)}$ is $(j_1, \ldots, j_s)$-strong.*

Proof of Claim 4. Since $|\mathcal{F}_{z,\vec{a}}| \geq [(sk)^{sk} \cdot f(k,L)^2]^{sk}$, by Lemma 3.3, there exist a tuple of distinct vertices $(y_1, \ldots, y_s)$ and and a tuple $(j_1, \ldots, j_s) \in \mathcal{A}$ such that the family

$$\mathcal{T} := \{F_{(j_1,\ldots,j_s)} : F \in \mathcal{F}_{z,\vec{a}} \text{ and } F_{(j_1,\ldots,j_s)} \text{ has leaf vector } (y_1, \ldots, y_s)\}$$

contains at least $p := (sk)^{sk-j} f(k,L)$ internally disjoint members $T_1, \ldots, T_p$, where

$j = j_1 + \cdots + j_s$. Since each member of $\mathcal{F}_{z,\vec{a}}$ is a feasible spider, each T_i is a feasible spider. Since $T_1, \ldots, T_p$ have the same leaf vector and are internally disjoint and $p \geq (sk)^{sk-j} f(k, L)$, by Definition 2.5, $\forall i \in [p]$, T_i is $(j_1, \ldots, j_s)$-strong. Also by the definition of $\mathcal{T}$, $\forall i \in [s]\, \exists\, F^{(i)} \in \mathcal{F}_{z,\vec{a}}$ such that $F^{(i)}_{(j_1,\ldots,j_s)} = T_i$.

It remains to show that $(j_1, \ldots, j_s) \in \mathcal{A} \setminus \mathcal{L}$. For that, it suffices to show $j_1 \notin \{1, k\}$. For each $i \in [p]$, let w_i denote the center of T_i, u_i its neighbor on the leg to y_1, and P_i the portion of the the leg from u_i to y_1. Then $w_i \in W_{\vec{a}}$ and $u_i \in U^-_{\vec{a}}$.

First, suppose that $j_1 = 1$. Then $u_1 = u_2 = \cdots = u_p = y_1$ but $w_1, \ldots, w_p$ are distinct, which yields $d_{H_{\vec{a}}}(u_1) \geq p = e(S) \geq (sk)^{sk-j} f(k, L) \gg 2kt$, which contradicts our definition of $U^-_{\vec{a}}$.

Now suppose that $j_1 = k$. Then $y_1 = z$. Since $T_1, \ldots, T_t$ are internally disjoint, $\bigcup_{i=1}^t P_i$ is a t-legged spider of height $k-1$ with center z and leaves $u_1, \ldots, u_t$. By (3.12), for each $i \in [t]$, $(u_i, a_2, \ldots, a_s)$ is $(1, k, \ldots, k)$-strong. Thus we can greedily find t spiders $Z_1, \ldots, Z_t$ with length vector $(1, k, \ldots, k)$, satisfying that every Z_i has leaf vector $(u_i, \vec{a})$ and that $Z_1 \cup P_1, \ldots, Z_t \cup P_t$ are t internally disjoint spiders with length vector $(k, \ldots, k)$ and leaf vector $(z, \vec{a})$. The union of these forms a copy $K^k_{s,t}$, a contradiction. This completes the proof of Claim 4. ∎

Claim 4 contradicts condition 2 of our assumption about $\mathcal{F}$. This completes the proof. ∎

3.3. Proof of Theorem 1.1. The main idea of the proof of Theorem 1.1 is roughly as follows. In an almost regular graph with minimum degree $\delta \geq \Omega(n^{\frac{1}{k} - \frac{1}{ks}})$ there are $\Omega(n\delta^{ks}) \geq \Omega(n^s)$ s-legged spiders of height k, that is, copies of $K^k_{1,s}$. Using the lemmas in the previous subsection as well as some new ones specific to the $k = 3, 4$ cases, we argue that most of these spiders do not contain critical paths of length at most k or any strong subspiders. Using the pigeonhole principle, we can find an s-tuple that is the leaf vector of a large number of $K^k_{1,s}$ that do not contain strong subspiders or critical paths of length at most k. This allows us to find t copies that are internally disjoint, whose union then forces a copy of $K^k_{s,t}$.

We need the following lemma, which holds only for $k = 3, 4$.

LEMMA 3.5. *Suppose that F is an $(\ell_1, \ldots, \ell_s)$-strong spider where $\forall i \in [s], 1 \leq \ell_i \leq k$. Then $\forall 1 \leq i < j \leq s$, $\ell_i + \ell_j \geq k + 1$. Moreover, if $k \in \{3, 4\}$, then either $\forall i \in [s]\, \ell_i \geq \frac{k}{2}$ or $(\ell_1, \ldots, \ell_s) \in \mathcal{L}$, where $\mathcal{L}$ is as defined in Lemma* 3.4.

Proof. Since F is strong and in particular is feasible, any subpath of F of length at most k is light. Suppose on the contrary that $\exists i, j$ with $\ell_i + \ell_j \leq k$. Without loss of generality, suppose $i = 1, j = 2$. Let $(v_1, \ldots, v_s)$ be the leaf vector of F. Let $\ell = \ell_1 + \cdots + \ell_s$. Since F is strong, by Definition 2.5, there are at least $(sk)^{sk-\ell} \cdot f(k, L) > f(k, L)$ internally disjoint feasible spiders with leaf vector $(v_1, \ldots, v_s)$ and length vector $(\ell_1, \ldots, \ell_s)$. In particular, in their union, there exist at least $f(k, L) \geq f(\ell_1 + \ell_2, L)$ internally disjoint light paths of length $\ell_1 + \ell_2$ joining v_1 and v_2. This means that the path in F that joins v_1 and v_2 is not $(\ell_1 + \ell_2)$-light, contradicting our earlier discussion.

Now assume that $k \in \{3, 4\}$ and there exists $i \in [s]$ with $\ell_i < \frac{k}{2}$. Fix such an i. Then $\ell_i = 1$. Since $\ell_i + \ell_j \geq k + 1$ for all $j \neq i$, we must have $\ell_j = k$ for all $j \neq i$. Hence $(\ell_1, \ldots, \ell_s) \in \mathcal{L}$. ∎

By Corollary 3.2 and Lemmas 3.4 and 3.5, for $k \in \{3, 4\}$ we have the following.

COROLLARY 3.6. *Let k, K, s, t be positive integers where $k \in \{3, 4\}$, $K \geq 1$, and $s \geq 3, t \geq 2$. Then provided that L is sufficiently large compared to s, t, k, and K, for* *any $\zeta > 0$ there exist $C, n_0 > 0$ such that the following holds. Suppose that G is a $K_{s,t}^k$-free K-almost-regular graph $n \geq n_0$ vertices with minimum degree $\delta \geq Cn^{\frac{1}{k}-\frac{1}{sk}}$. Let $\mathcal{F}$ denote the family of feasible s-legged spiders of height k in G that contain a strong s-legged subspider. Then $|\mathcal{F}| \leq \zeta n\delta^{sk}$.*

Now, we are finally ready to prove our main theorem.

Proof of Theorem 1.1. First we set some constants. Fix integers $s, t \geq 2$ and $k \in \{3, 4\}$. Let K be obtained by Lemma 2.1 with $\epsilon = \frac{1}{k} - \frac{1}{sk}$. Choose L to be a large constant such that Lemma 2.4 and Corollary 3.6 are valid. We further require that L is large enough such that $\frac{2(sk)!K^{sk}}{f(1,L)} \leq \frac{1}{4k(sk+1)!}$. Let $C_1 = [(sk)^{sk} \cdot f(k, L)^2]^{sk}$, that is, the constant in Lemma 3.3 with $\ell = sk$. Let C be a large constant such that Corollary 3.6 holds with $\zeta := \frac{1}{8(sk+1)!}$. We further require that C is large enough such that $\frac{C^{sk}}{8(sk+1)!} \geq C_1$.

By Lemma 2.1, it suffices to show the following statement. For sufficiently large n, if G is an n-vertex K-almost-regular graph with minimum degree $\delta \geq Cn^{\frac{1}{k}-\frac{1}{sk}}$, then G contains a copy of $K_{s,t}^k$.

We will prove this by contradiction. Suppose to the contrary that G is $K_{s,t}^k$-free. Let $\mathcal{F}$ be the family of all the s-legged spiders of height k in G. Then by a greedy process, it is easy to see that

$$|\mathcal{F}| \geq \frac{1-o(1)}{(sk+1)!} n\delta^{sk} \geq \frac{n\delta^{sk}}{2(sk+1)!}, \tag{3.14}$$

where the last inequality holds because n is sufficiently large. By Lemma 2.4, for every $2 \leq \ell \leq k$, the number of ℓ-critical paths is at most $\frac{2n(K\delta)^\ell}{f(\ell-1,L)}$. Since the maximum degree of G is at most $K\delta$, the number of members of $\mathcal{F}$ that contain a ℓ-critical path is at most $\binom{sk}{\ell}\frac{2n(K\delta)^\ell}{f(\ell-1,L)} \cdot (K\delta)^{sk-\ell} \leq \frac{2(sk)!K^{sk}}{f(\ell-1,L)} n\delta^{sk} \leq \frac{2(sk)!K^{sk}}{f(1,L)} n\delta^{sk} \leq \frac{n\delta^{sk}}{4k(sk+1)!}$, where the factor $\binom{sk}{\ell}$ upper bounds the number of positions of an ℓ-critical path in the s-legged spider of height k, and the last inequality holds by the choice of L. So the number of members of $\mathcal{F}$ that contain a critical path of length at most k is no more than $(k-1) \cdot \frac{n\delta^{sk}}{4k(sk+1)!} < \frac{n\delta^{sk}}{4(sk+1)!}$. Let $\mathcal{F}'$ denote the family of feasible members of $\mathcal{F}$. It follows that

$$|\mathcal{F}'| \geq |\mathcal{F}| - \frac{n\delta^{sk}}{4(sk+1)!} \geq \frac{n\delta^{sk}}{4(sk+1)!}, \tag{3.15}$$

where in the last inequality we used (3.14).

Let $\mathcal{F}''$ denote the family of members of $\mathcal{F}'$ that contain no strong s-legged subspider. By Corollary 3.6 we have that

$$|\mathcal{F}' \setminus \mathcal{F}''| \leq \zeta n\delta^{sk} = \frac{n\delta^{sk}}{8(sk+1)!},$$

where the last equality holds by the choice of ζ. This, together with (3.15), gives us that

$$|\mathcal{F}''| = |\mathcal{F}'| - |\mathcal{F}' \setminus \mathcal{F}''| \geq \frac{n\delta^{sk}}{4(sk+1)!} - \frac{n\delta^{sk}}{8(sk+1)!} = \frac{n\delta^{sk}}{8(sk+1)!}.$$

Since $\delta \geq Cn^{\frac{1}{k}-\frac{1}{sk}}$, it follows that

$$|\mathcal{F}''| \geq \frac{C^{ks}}{8(sk+1)!} n^s \geq C_1 n^s,$$

where the last inequality holds because of the choice of C. By averaging, there exists a tuple $(v_1, \ldots, v_s)$ of distinct vertices such that the subfamily $\mathcal{F}_1$ of $\mathcal{F}''$ that consists of all the members of $\mathcal{F}''$ that have leaf vector $(v_1, \ldots, v_s)$ has size at least

$$|\mathcal{F}_1| \geq \frac{|\mathcal{F}''|}{n^s} \geq \frac{C_1 n^s}{n^s} = C_1.$$

Now $\mathcal{F}_1$ is a family of s-legged feasible spiders that have the same leaf vector and length vector. Since $|\mathcal{F}_1| \geq C_1$, by Lemma 3.3 and our choice of C_1 given at the beginning of this proof, there exists a member of $\mathcal{F}''$ containing a strong s-legged subspider. This contradicts our definition of $\mathcal{F}''$ and completes the proof. ∎

4. Concluding remarks. It is easy to derive from our discussions that the following weakening of Conjecture 1.1 holds.

PROPOSITION 4.1. *Let $s, t, k \geq 2$ be integers. Let $\mathcal{K}_{s,t}^{\leq k}$ denote the family of graphs that can be be obtained from $K_{s,t}$ by replacing each edge uv with a path of length at most k between u and v so that the st replacing paths are internally disjoint. Then* $\mathrm{ex}(n, \mathcal{K}_{s,t}^{\leq k}) = O(n^{1+\frac{1}{k}-\frac{1}{sk}})$.

This together with the general theorem of Bukh and Conlon [2] implies the following.

COROLLARY 4.2. *Let $s, k \geq 2$ be integers. Then for sufficiently large t,* $\mathrm{ex}(n, \mathcal{K}_{s,t}^{\leq k}) = \Theta(n^{1+\frac{1}{k}-\frac{1}{sk}})$.

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Since our manuscript was submitted, Janzer [18] has extended our work and proved Conjecture 1.1 for all k.

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