Supplementary Material

for

On Using the Hypervolume Indicator to Compare Pareto Fronts: Applications to Multiple Optimal Experiment Design by Cao, Smucker, and Robinson

Lemma 3. The hypervolume indicator, $I_H(\mathrm{PF}, \mathbf{r})$, for a 3-dimensional discrete Pareto front in $[0,1]^3$ of size $p \geq 3$ with respect to a reference point $\mathbf{r} = (r_1 \leq 0, r_2 \leq 0, r_3 \leq 0)$ can be given by an expression of the form

$$\sum_{i=1}^{s} \left[\left(\sum_{j=1}^{p_i} f_1(\xi_{ij})\, f_2(\xi_{ij}) - \sum_{j=1}^{p_i-1} f_1(\xi_{ij})\, f_2(\xi_{ij+1}) + |r_1|\, f_2(\xi_{i1}) + |r_2|\, f_1(\xi_{ip_i}) + r_1 r_2 \right) \times (f_{3,i} - f_{3,i-1}) \right],$$

where s is number of slices, p_i is the number of non-dominated points in the remaining two criteria that contained by the i^{th} slice, $f_{3,i}$ is the i^{th} distinct criterion 3 value , $f_{3,0} = r_3$ and $f_{3,i} > f_{3,i-1}$ for $i = 1, \cdots, s$.

Proof. Suppose criterion 3, f_3, can be classified into s 3-dimensional slices. Suppose the i^{th} ($i = 1, \cdots, s$) slice consists of p_i non-dominated solutions in the remaining two criteria (Note a slice may contain more than p_i points, however, points are dominated in whatever criteria remain will contribute nothing to the volume of that slice). Then by Lemma 1 the area (hypervolume) of its base is $\sum_{j=1}^{p_i} f_1(\xi_{ij})\, f_2(\xi_{ij}) - \sum_{j=1}^{p_i-1} f_1(\xi_{ij})\, f_2(\xi_{ij+1}) + |r_1|\, f_2(\xi_{i1}) + |r_2|\, f_1(\xi_{ip_i}) + r_1 r_2$, and its height is given by $f_{3,i} - f_{3,i-1}$. If we have ordered the slices such that $f_{3,i} > f_{3,i-1}$ for $i = 1, \cdots, s$, then $f_{3,1}$ will be the criterion 3 value for the bottom slice, and $f_{3,0} = r_3$ by assumption. Thus we have

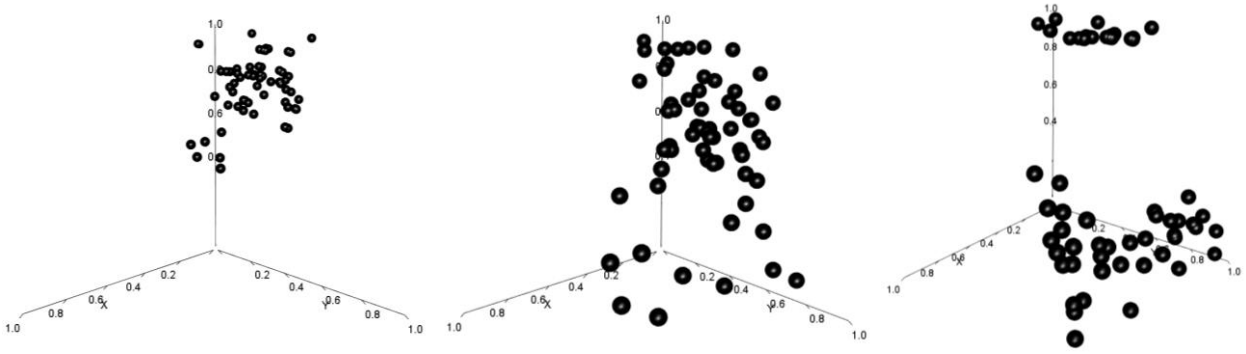
$$I_H(\mathrm{PF}, \mathbf{r}) = \sum_{i=1}^{s} \left[\left(\sum_{j=1}^{p_i} f_1(\xi_{ij})\, f_2(\xi_{ij}) - \sum_{j=1}^{p_i-1} f_1(\xi_{ij})\, f_2(\xi_{ij+1}) + |r_1|\, f_2(\xi_{i1}) + |r_2|\, f_1(\xi_{ip_i}) + r_1 r_2 \right) \times (f_{3,i} - f_{3,i-1}) \right]$$

. □

The Three Pareto fronts, PF_{61}, PF_{61R} and PF_{61E}, in Section 4.3.

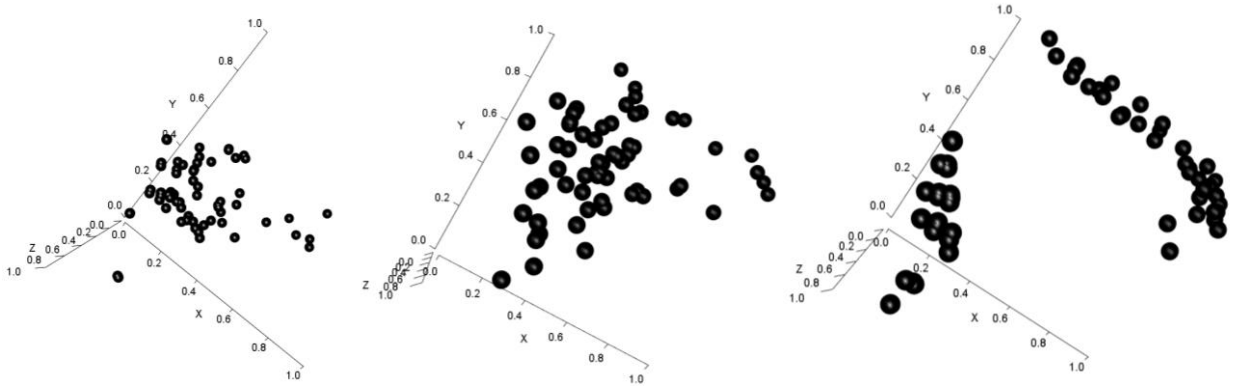
(1) Front view

PF_{61} PF_{61R} PF_{61E}



(2) Top view

PF_{61} PF_{61R} PF_{61E}



(3) Back view

PF_{61} PF_{61R} PF_{61E}