

Upper Bounds for Chebyshev Permutation Arrays

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Abstract: We improve on known upper bounds for the size of permutation arrays under the Chebyshev metric, defined as follows. The Chebyshev distance between permutations π and σ on the symbols $\{1, 2, \dots, n\}$, denoted by $d(\pi, \sigma)$, is $\max\{|\pi_i - \sigma_i| \mid 1 \leq i \leq n\}$. For an array A (set) of such permutations, the Chebyshev distance of A , denoted by $d(A)$, is $\min\{d(\pi, \sigma) \mid \pi, \sigma \in A, \pi \neq \sigma\}$. An array A of such permutations with $d(A) = d$ will be called an (n, d) -PA. Let $P(n, d)$ denote the maximum size of any (n, d) -PA. The function $P(n, d)$ has been the subject of previous research. In this paper, we consider strings on the symbols $\{0, 1, 2\}$, with the 0's representing low symbols and the 2's high symbols for the function $P(n, d)$. An array A of such strings of length n is *separable* if for any two strings in A , there is a position $1 \leq i \leq n$ such that the i^{th} symbol in one string is 0 and the i^{th} symbol in the other is a 2. The maximum size of a separable array of strings of length n , with a occurrences of the symbol 0 and b occurrences of the symbol 2, is denoted by $R(n; a, b)$. We show that $R(n; k, k)$ is an upper bound for $P(n, n - k)$ when $k \leq \frac{n}{2}$. We derive upper bounds for $R(n; a, b)$ by various recursive and combinatorial methods, from which follow upper bounds for the Chebyshev function $P(n, d)$, which improve upon previous such upper bounds in the literature.

Keywords: permutation arrays; Chebyshev metric; upper bounds



Academic Editor: Patrick Solé

Received: 22 April 2025

Revised: 20 May 2025

Accepted: 20 May 2025

Published: 26 May 2025

Citation: Bereg, S.; Miller, Z.; Sudborough, I.H. Upper Bounds for Chebyshev Permutation Arrays.

Entropy **2025**, *27*, 558. <https://doi.org/10.3390/e27060558>

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1. Introduction

In refs. [1,2], studies of permutation arrays under the Chebyshev metric were presented. This complemented many studies of permutation arrays under other metrics, such as the Hamming metric [3–5], Kendall τ metric [6,7], and several others [8]. The use of the Chebyshev metric was motivated by applications of error correcting codes and recharging in flash memories [6].

The flash memory application is based on a rank-modulation scheme [9], which eliminates the need to use absolute values of cell levels in storing information. Instead, relative ranks are used. The data are coded by permutations of a finite number of ranks.

Let $\pi = \pi_1\pi_2 \dots \pi_n$ and $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ be two permutations on the symbols in $\{1, 2, \dots, n\}$. The Chebyshev distance between π and σ , denoted by $d(\pi, \sigma)$, is $\max\{|\pi_i - \sigma_i| \mid 1 \leq i \leq n\}$. For an array (set) of permutations, say, A , the Chebyshev distance of A , denoted by $d(A)$, is $\min\{d(\pi, \sigma) \mid \pi, \sigma \in A, \pi \neq \sigma\}$. An array A of permutations on $\{1, 2, \dots, n\}$ with $d(A) = d$ will be called an (n, d) -PA. Let $P(n, d)$ denote the maximum size of any (n, d) -PA. We shall also define analogously the Chebyshev distance between two strings and the Chebyshev distance of an array of strings. The context will make clear whether the objects are strings or permutations.

Previous work on permutation arrays under the Chebyshev metric gave upper bounds based on a Gilbert–Varshamov inequality [10,11] (see our Theorem 2, or, for a recursive inequality, see our Theorem 3). In [2], it was also shown for fixed $r \geq 1$ that there exist constants c_r and d_r such that $P(d + r, d) = c_r$ for $d \geq d_r$ (see our Theorem 1). Upper bounds on c_r and d_r were given in [2]. We give substantial improvements on these upper bounds.

We consider strings over the alphabet $\{0, 1, 2\}$. A set A of such strings of length n is *separable* if for any two strings in A , there is a position $1 \leq j \leq n$ such that the j^{th} symbol in one string is 0 and the j^{th} symbol in the other is a 2. We will often view such a set A as a matrix in which the rows are the strings in A (ordered arbitrarily) and the columns are the coordinate positions $1, 2, \dots, n$ of entries in these strings. So, the (i, j) 'th entry of A in this view is the entry $(0, 1, \text{ or } 2)$ in the j^{th} position of string i of A . If every string in a separable array A has length n and has a occurrences of the symbol 0 and b occurrences of the symbol 2, then we call A an (n, a, b) -array. The maximum number of strings in an (n, a, b) -array is denoted by $R(n; a, b)$. Examples of a $(5, 2, 2)$ array and a $(7, 1, 3)$ array are shown in Figure 1.

Such matrices, consisting of the three symbols 0, 1, and 2, are reminiscent of weighing matrices. A *weighing matrix* W of weight w is a square matrix of rank n containing symbols $-1, 0,$ and 1 such that $W \cdot W^T = wI_n$ [12]. A weighing matrix is an extension of Hadamard matrices [13] by adding the symbol 0 (see [14]). A *circulant weighing matrix* is a weighing matrix in which each row is a circular shift of the first row [14]. There are examples of $n \times n$ weighing matrices that can be transformed to an (n, a, b) -array, for appropriate a and b , by transforming the three symbols $(-1, 0, 1)$ to $(0, 1, 2)$, respectively. For example, the circulant weighing matrix with first row $(-1\ 1\ 0\ 1\ 1\ 0)$ is transformed into a circulant $(6, 1, 3)$ -array with first row $(0\ 2\ 1\ 2\ 2\ 1)$ by the indicated replacement of symbols.

The motivation for our study begins with Lemma 1, given in the next section, showing that any upper bound for $R(n; k, k)$ also serves as an upper bound for the Chebyshev function $P(n, n - k)$. The resulting upper bounds we obtain on the Chebyshev function give improvements over what was previously known in the literature.

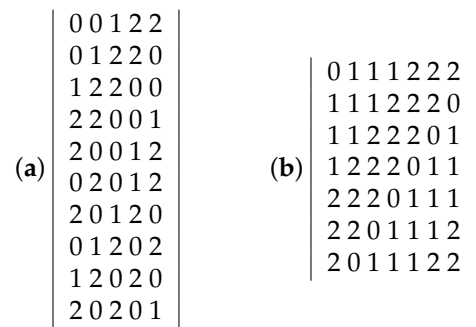


Figure 1. (a) $(5,2,2)$ -array with 10 rows and (b) $(7,1,3)$ -array with 7 rows.

Our results are the following.

1. By means of a transformation, we adapt a result of Bollobás from the theory of extremal sets to show that $R(n; s, t) \leq \binom{2s+2t}{s+t}$. From this, we derive $P(d + r, d) \leq \binom{4r}{2r} = \frac{2^{4r}}{\sqrt{2\pi r}}(1 + o(1))$ (as r grows). This improves on the previously known upper bound for $P(d + r, d)$ for large r .
2. We develop recursive methods for upper bounding $R(n; s, t)$. For small r , these yield upper bounds for $P(d + r, d)$, which improve on both the previously known upper bounds for $P(d + r, d)$ and on the bound obtained through the transformation of the Bollobás result stated above.

We will need the following notation and terms. For an array A , we let $|A|$ be the number of rows in A . For any row r in an array A , we let $r(i)$ be the entry of r in column i

(which we also call *position i*) of A . Given a set S of rows in A , we say that S is *separated in a set of columns C* if for any two rows $r, s \in S$ there is a column $i \in C$ such that one of $\{r(i), s(i)\}$ is 0 while the other is 2. Given two sets S_1, S_2 of rows in A , we refer to *internal separations* of S_1 and S_2 as that set of columns at which either S_1 is separated or S_2 is separated. We refer to *cross separations* of S_1 and S_2 as that set of columns C such that for any two rows $r \in S_1 \setminus S_2$ and $s \in S_2 \setminus S_1$, there is a column $c \in C$ at which $\{r, s\}$ are separated.

2. Background and Preliminary Results

We begin with a theorem by Klove et al. [2], preceded by a definition.

Definition 1. *If A is a $(d + r, d)$ -PA, then the integers $1, 2, \dots, r$ and $d + 1, d + 2, \dots, d + r$ are called *potent symbols*. Moreover, the integers $1, 2, \dots, r$ are called *low potent symbols* and the integers $d + 1, d + 2, \dots, d + r$ are called *high potent symbols*.*

Their proof of the following upper bound, here omitted, uses the idea of potent symbols.

Theorem 1 (Klove et al. [2]). *For fixed $r \geq 1$, there exist constants c_r and d_r such that $P(d + r, d) = c_r$ for $d \geq d_r$. Moreover,*

$$c_r \leq 2^{2^r} (2r)! \tag{1}$$

and

$$d_r \leq 1 + (2r - 1)c_r - r. \tag{2}$$

Exact values are known for $r = 1$ and $r = 2$, namely, $c_1 = 3, d_1 = 2$ [2], $c_2 = 10$, and $d_2 = 3$ [1]. The upper bounds for c_r and d_r given in (1) and (2) turn out to be quite generous. For example, inequality (1) gives the bound $c_2 \leq 384$, but we know that $c_2 = 10$. We will see additional improvements on c_r as given in Equation (1) later in this paper.

The role of potent symbols motivates the idea behind the following lemma, which establishes the connection between the Chebyshev function and $R(n; k, k)$. Consider an $(n, n - k)$ -PA, which we call A , and any row π of A . The idea is to put the symbols $1, 2, \dots, n$ of π into three groups. Those that are high (resp. low) potent symbols, namely, $n - k + 1, n - k + 2, \dots, n$ (resp. $1, 2, \dots, k$), are relabeled 2 (resp. 0), while all other symbols are relabeled 1. Repeating this replacement over all rows of A , independently in any two rows, will yield an (n, k, k) -array, as we will see below. On the other hand, given an (n, k, k) array B , we perform the inverse replacement independently in each row of B to obtain an $(n, n - 2k + 1)$ -PA. We obtain bounds linking the R function with the Chebyshev function in the lemma following.

Lemma 1. $P(n, n - k) \leq R(n; k, k) \leq P(n, n - 2k + 1)$ when $k \leq \frac{n}{2}$.

Proof. We begin with the first inequality. Let A be an $(n, n - k)$ -PA, where $k \leq \frac{n}{2}$. Create an (n, k, k) -array A' as follows. For each row $\pi \in A$, create a row $\pi' \in A'$ by

$$\pi'(i) = \begin{cases} 0 & \text{if } \pi(i) \in \{1, \dots, k\}, \\ 2 & \text{if } \pi(i) \in \{n - k + 1, \dots, n\}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then, A' is an array over the symbols $\{0, 1, 2\}$, having k many 0's and k many 2's in each row. Since $d(A) \geq n - k$, then for any two rows π and σ of A , there is a position i such

that $|\pi(i) - \sigma(i)| \geq n - k$. So, one of $\pi(i)$ or $\sigma(i)$ is $\geq n - k + 1$ and the other must be $\leq k$. Consequently, one of $\pi(i)$ or $\sigma(i)$ is transformed into a 2 and the other into a 0. So the rows π' and σ' are separated in A' . Furthermore, as any two rows of A' are separated, any two such rows must be distinct. Therefore A' is an (n, k, k) -array and $R(n; k, k) \geq |A'| = |A|$, so the inequality follows.

Consider now the second inequality. Let B be an (n, k, k) -array with the maximum possible number $R(n; k, k)$ of rows. Let B' be the permutation array obtained from B by arbitrarily replacing, in any row of B , the k many 2's by the high potent symbols $n - k + 1, n - k + 2, \dots, n$, the k many 0's by the low potent symbols $1, 2, \dots, k$, and the 1 symbols by the symbols $k + 1, \dots, n - k$. (It is, of course, required that the replacements create a permutation). The replacements performed on any two rows of B are performed independently of each other. Since B is separable, given any two rows r and s of B' , there is a column c in B' for which one of $r(c), s(c)$ is a high potent symbol while the other is a low potent symbol. So, we have $|r(c) - s(c)| \geq n - 2k + 1$. Since rows r and s were arbitrary, this shows that $d(B') \geq n - 2k + 1$. So, by monotonicity in d of $P(n, d)$, we obtain $P(n, n - 2k + 1) \geq |B'| = |B| = R(n, k, k)$. \square

The following Corollary of Lemma 1 shows that $R(n; k, k)$ reaches a maximum that depends only on k .

Corollary 1. *There are constants n_k and m_k (depending only on k) such that for all $n \geq n_k$, we have $R(n; k, k) \leq m_k$. Moreover, we can take $m_k = c_{2k-1}$ and $n_k = 2k + d_{2k-1} - 1$. Here, c_{2k-1} and d_{2k-1} are the constants from Theorem 1.*

Proof. By the second inequality of Lemma 1, we have $R(n; k, k) \leq P(n, n - 2k + 1) = P(n - 2k + 1 + (2k - 1), n - 2k + 1)$. So, by Theorem 1, $P(n, n - 2k + 1) \leq c_{2k-1}$ for $n - 2k + 1 \geq d_{2k-1}$; that is, $R(n; k, k) \leq c_{2k-1}$ for $n \geq 2k + d_{2k-1} - 1$. \square

We note that a transformation of an (n, k, k) -array with N rows into an $(n, n - k)$ -PA with N rows is not known to be always possible.

We will see later that the existence of the constants n_k and m_k follows from one of our theorems (Theorem 6), together with an improvement on the bounds given in Theorem 1 for the constants c_r and d_r . Still, we mention Corollary 1 here to show that the existence of n_k and m_k is already implied by Theorem 1 combined with the argument in that Corollary.

There are a few other theorems in the literature that give upper bounds on the Chebyshev function. Let $V(n, d)$ be the number of permutations on $\{1, 2, \dots, n\}$ within Chebyshev distance d of the identity permutation.

Theorem 2 (Theorem 11 [2]). *For even d and $2d \geq n \geq d \geq 2$, $P(n, d) \leq \frac{(n+1)!}{V(n+1, d/2)}$.*

Theorem 3 (Theorem 12 [1]). *For $1 \leq k \leq d < n$,*

$$P(n, d) \leq P(n - k, d) \cdot \binom{n}{k}.$$

Corollary 2. *For $s \leq t$ and $1 \leq k \leq s$,*

$$R(n; s, t) \leq R(n - k; s - k, t) \cdot \binom{n}{k}.$$

Proof. Consider any (n, s, t) -array A , which we can take to be of maximum possible size $R(n; s, t)$. Create subsets of the rows of A , determined by the positions of their k many 0's. That is, two rows are in the same subset if they both have k many 0's occurring in the

same k positions. There are $\binom{n}{k}$ such sets. Any two rows in such a set must be separated somewhere in the remaining $n - k$ positions, using $s - k$ many 0's and t many 2's in those $n - k$ positions. Hence, any such set of rows must have the size at most $R(n - k; s - k, t)$. It follows that $|R(n; s, t)| = |A| \leq R(n - k; s - k, t) \cdot \binom{n}{k}$. \square

Corollary 3. For all $2 \leq t \leq n - 2$, $R(n; 2, t) \leq \binom{n}{2}$.

Proof. Setting $s = k = 2$ in Corollary 2, we have $R(n - 2, s - k, t) = R(n - 2; 0, t) = 1$ since $R(m, 0, t) = 1$ for all m . \square

We will see later (Theorem 9) a bound on $R(2, t)$ that depends only on t once n is big enough. But, the bound in Corollary 3 is still the best for n that is small enough relative to t , as we will see.

Some of the best previous upper bounds for small n were given using Theorem 3. For example, from Theorem 3, $P(n, n - 3) \leq \min\{\binom{n}{3}, 3\binom{n}{2}, 10\binom{n}{1}\}$, choosing $k = 3, 2, 1$, respectively. To see this, consider the following. Since $P(r, r) = 1$ for any r , taking $k = 3$, we obtain $P(n - 3, n - 3) = 1$. As mentioned previously after Theorem 1, $c_1 = 3$, so, taking $k = 2$, we obtain $P(n - 2, n - 3) = 3$. Again, recalling $c_2 = 10$, we take $k = 1$ to obtain $P(n - 3, n - 1) = 10$. Since $\min\{\binom{n}{3}, 3\binom{n}{2}, 10\binom{n}{1}\} = 10n$ for all $n \geq 10$, Theorem 3 gives the upper bound $P(n, n - 3) \leq 10n$. In an application of our recursive upper bounds, we will see later in Corollary 7 that $R(n; 3, 3) \leq 169$, yielded by Lemma 1. $P(n, n - 3) \leq R(n; 3, 3) \leq 169$. Thus, Theorem 3 gave the best upper bound for $P(n, n - 3) \leq 10n$ for $n \leq 16$, while our new recursive results give an improved bound for $P(n, n - 3) \leq 169$ when $n \geq 17$.

Similarly, from Theorem 3, $P(n, n - 4) \leq \min U$, where $U = \{\binom{n}{4}, 3\binom{n}{3}, 10\binom{n}{2}, P(n, n - 3)\binom{n}{1}\}$, choosing $k = 4, 3, 2, 1$, respectively. The previous paragraph shows that $P(n, n - 3) \leq 169$ for all $n \geq 17$. Calculating, $\min U = 10\binom{n}{2}$ for $14 \leq n \leq 34$ and $= 169n$ for all $n \geq 35$. In Corollary 8, which we will see later, we obtain $R(n; 4, 4) \leq 3087$. Calculating, one observes that $3087 \leq \min U$ for all $n \geq 19$. So, we obtain the improved upper bound: $P(n, n - 4) \leq R(n; 4, 4) \leq 3087$ for all $n \geq 19$.

We observe that $R(n; a, b)$ is symmetric and monotone; that is,

$$R(n; a, b) = R(n; b, a), \tag{3}$$

$$R(n; a, b) \geq R(m; a, b) \text{ if } n \geq m. \tag{4}$$

Later in this paper, it will be useful to consider separable arrays on $\{0, 1, 2\}$ in which the number of 0's and 2's in each row is not constant for all rows. The following definition and the lemma which follows treat this case.

Definition 2. For $a, b \geq 2$, an $(n, \leq a, \leq b)$ -array is a separable array of length n strings over $\{0, 1, 2\}$ such that each string in A has at most a many 0's and at most b many 2's. Let $R(n; \leq a, \leq b)$ be the maximum size of any $(n, \leq a, \leq b)$ -array.

Lemma 2. If $s, t \geq 1$ and $n \geq s + t$, then

$$R(n; \leq s, \leq t) \leq R(n; s, t). \tag{5}$$

Proof. Let A be an $(n, \leq s, \leq t)$ -array with entries from $\{0, 1, 2\}$, realizing $R(n; \leq s, \leq t)$. Consider any string π in A having $s' \leq s$ many 0's and $t' \leq t$ many 2's. Since $n - s - t \geq 0$, we have $n - s' - t' \geq s - s' + t - t'$, so A must have at least $s - s' + t - t'$ many 1's in its row corresponding to π . We transform π into a string π' with s many 0's and t many 2's as follows. We convert any $s - s'$ of the 1's in π to 0's and any $t - t'$ of the remaining 1's to 2's

and let π' be the resulting string. Let A' be the array obtained from A by replacing each $\pi \in A$ by π' .

It suffices to show that A' is separable. It would then follow that no two strings $\pi', \sigma' \in A'$ are the same, since they would not be separated, and since the number of rows in A' is at most $R(n; s, t)$, then, also, $R(n; s', t') \leq R(n; s, t)$. So, let π, σ be two arbitrary strings of A and let π', σ' be the respective transformed strings in A' . There is a column c of A where $\{\pi(c), \sigma(c)\} = \{0, 2\}$ (in either order). Since the transformation affects no symbols in π or σ that are either 0 or 2, it follows that $\pi'(c) = \pi(c)$ and $\sigma'(c) = \sigma(c)$ and, hence, π' and σ' are separated. Thus, A' is separable. \square

The following theorem will lead to the exact value $R(n; 2, 2) = 10$ for $n \geq 5$.

Theorem 4. *Suppose that $R(n_0; k, k) \leq m$ such that*

$$2k(m + 1) < (n_0 + 1)(1 + \lfloor n_0 / (2k - 1) \rfloor). \tag{6}$$

Then, $R(n; k, k) \leq m$ for all $n \geq n_0$.

Proof. Suppose to the contrary that $R(n; k, k) \geq m + 1$ for some $n > n_0$. Let n be the smallest such number. Let $A = \{\pi_1, \pi_2, \dots, \pi_{m+1}\}$ be an (n, k, k) -array. Let k_i denote the number of 0 and 2 symbols in position i , taken over all rows of A . Let $z = 1 + \lfloor n_0 / (2k - 1) \rfloor$, so that $n_0 \geq (z - 1)(2k - 1)$. We show that $k_i \geq z$ for all i . Suppose, by symmetry of argument, that $k_1 \leq z - 1$ and (by rearranging the order) only $\pi_i, 1 \leq i \leq k_1$, have 0 or 2 symbols in the first position. By our assumption, all of the first k_1 rows, and only the first k_1 rows, have a 0 or 2 symbol in position 1. So, if there are $z - 1$ rows, each adding $2k - 1$ 0 or 2 symbols to some position $j > 1$, the total number of 0 or 2 symbols (other than the one in position 1) is $(2k - 1)(z - 1)$. Since the number of positions, namely, $n > n_0$, is greater than $(2k - 1)(z - 1)$, by the pigeonhole principle, there is a position $j > 1$ where no $\pi_i, 1 \leq i \leq k_1$ do not has any 0 or 2 symbols. Now, do the following:

- For each row $\pi_i, 1 \leq i \leq k_1$, exchange the 0 or 2 symbol in position 1 with the symbol in position j .
- Delete the symbol in position 1 in all rows.

The result is a separable array of $\geq m + 1$ rows, where each row is a string of length $n - 1$. This contradicts our choice of n being the smallest. So, we have $k_i \geq z$ for all i .

Note that the total number of 0 and 2 symbols in the (n, k, k) -array A is $\geq 2k(m + 1)$. As $k_i \geq z$, for all i , we have $2k(m + 1) \geq nz \geq (n_0 + 1)(1 + \lfloor n_0 / (2k - 1) \rfloor)$, which contradicts inequality (6). So, the (n, k, k) -array A with $\geq m + 1$ rows does not exist. \square

Theorem 15 was used in [1] to prove that $P(n, n - 2) = 10$ for all $n \geq 5$. A similar proof shows that $R(n; 2, 2) = 10$ for all $n \geq 5$.

Corollary 4. *For all $n \geq 5, R(n; 2, 2) = 10$.*

Proof. $R(n; 2, 2) \geq 10$ for all $5 \leq n \leq 11$, by computation. In Theorem 4, set $n_0 = 11, k = 2$, and $m = 10$. Then, $z = 1 + \lfloor n_0 / (2k - 1) \rfloor = 4$ and $2k(m + 1) = 44 < 48 = (n_0 + 1)z$. So, $R(n; 2, 2) \leq 10$ for all $n \geq 11$, following Theorem 4.

Therefore, $R(n; 2, 2) = 10$ for all $n \geq 5$. \square

An example of a $(5, 2, 2)$ -array with 10 rows realizing $R(5; 2, 2)$ is given in Figure 1a .

3. Applying a Result of Bollobás

We begin with the following result of Bollobás from the theory of extremal sets. It is actually a reformulation, given in [15], of a theorem on saturated hypergraphs originally appearing in [16]. The proof can be found in [15].

Theorem 5. For two nonnegative integers a and b , write $w(a, b) = \binom{a+b}{a}^{-1}$. Let $\{(A_i, B_i : i \in I\}$ be a finite collection of finite sets such that $A_i \cap B_j = \emptyset$ if $i = j$. For $i \in I$, set $a_i = |A_i|$, $b_i = |B_i|$. Then, $\sum_{i \in I} w(a_i, b_i) \leq 1$ with equality if there is a set Y and integers a, b such that $0 \leq a \leq a + b \leq |Y|$, and $\{(A_i, B_i) : i \in I\}$ is the collection of all ordered pairs of subsets of Y with $|A_i| = a_i$ and $|B_i| = b_i$.

In particular, if $a_i = a$ and $b_i = b$ for all $i \in I$, then $|I| \leq \binom{a+b}{a}$.

We obtain an upper bound for $R(n; s, t)$ by reducing to the above theorem.

Theorem 6. $R(n; s, t) \leq \binom{2s+2t}{s+t}$.

Proof. Let M be a an (n, s, t) -array realizing $R(n; s, t)$, and set $p = R(n; s, t)$. For any $1 \leq i \leq p$, let $S_i = \{a_{i1} < a_{i2} < \dots < a_{is}\}$ be the set of column indices at which row i of M has 0 entries and $T_i = \{b_{i1} < b_{i2} < \dots < b_{it}\}$ the set of column indices at which row i of M has 2 entries.

Now, construct a $p \times 2n$ array Q whose first n columns are the same as in M and whose subarray M' consisting of the last n columns is obtained by interchanging 0 and 2 entries in M , leaving the 1 entries unchanged. That is, M' is obtained from M by flipping each 2 entry of M to a 0 entry in M' , flipping each 0 entry in M to a 2 entry in M' and leaving each 1 in M unchanged as a 1 entry in M' . Let $S'_i = \{a'_{ij} : 1 \leq j \leq s\}$ (resp. $T'_i = \{b'_{ij} : 1 \leq j \leq t\}$) be the column indices in M' corresponding to the column indices of S_i (resp. T_i) by a translation of n . That is, we have $a'_{ij} = a_{ij} + n$ and $b'_{ij} = b_{ij} + n$. Now, for each $i, 1 \leq i \leq p$, define two sets of column indices, Q_i and Q'_i , of Q by $Q_i = S_i \cup T'_i$ and $Q'_i = S'_i \cup T_i$. Observe that Q_i (resp. Q'_i) is the set of column indices at which row i of Q has 0 (resp. 2) entries.

We now show that for $1 \leq i \leq p$, the sets Q_i, Q'_i can play the roles of A_i and B_i (respectively) in the statement of Theorem 5 with $p = |I|$. Trivially, we have $Q_i \cap Q'_i = \emptyset$ for each i since $S_i \cap T_i = \emptyset$. Now, take $j \neq i, 1 \leq j \leq p$. We must show that $Q_i \cap Q'_j \neq \emptyset$ and $Q'_i \cap Q_j \neq \emptyset$. Since M is separable, we have $S_i \cap T_j \neq \emptyset$ or $S_j \cap T_i \neq \emptyset$, so assume by symmetry that $S_i \cap T_j \neq \emptyset$. Then, immediately, we have $Q_i \cap Q'_j \neq \emptyset$ since $S_i \subset Q_i$ and $T_j \subset Q'_j$. Also, it follows from $S_i \cap T_j \neq \emptyset$ and the interchange of 0's and 2's that $S'_i \cap T'_j \neq \emptyset$. Therefore, $Q'_i \cap Q_j \neq \emptyset$ since $S'_i \subset Q'_i$ and $T'_j \subset Q_j$. Thus, the conditions of Theorem 5 are satisfied. Since $|Q_i| = |Q'_i| = s + t$ for all i , we obtain $p = |I| \leq \binom{2s+2t}{s+t}$. \square

The preceding theorem implies Corollary 1 with the considerably improved value $m_k = \binom{4k}{2k}$ over that obtained by combining that corollary and Theorem 1.

Corollary 5. $P(d + r, d) \leq \binom{4r}{2r} = \frac{2^{4r}}{\sqrt{2\pi r}}(1 + o(1))$ (as r grows).

Proof. By Lemma 1 and Theorem 6, we have $P(d + r, d) \leq R(d + r; r, r) \leq \binom{4r}{2r}$. The final equality follows from the Stirling approximation applied to $\binom{4r}{2r}$. \square

We note that the preceding corollary implies that $c_r \leq \frac{2^{4r}}{\sqrt{2\pi r}}(1 + o(1))$, where c_r is the constant in Theorem 1. This is an improvement on the upper bound for c_r given in that theorem.

4. Recursive Techniques

4.1. The Positions Method

In this subsection, we introduce a technique, called *positions*, to recursively obtain an upper bound for $R(n; a, b)$. The strategy involves considering a fixed row π of a (n, a, b) -array A , with a occurrences of the symbol 0 in positions p_1, p_2, \dots, p_a and b occurrences of the symbol 2 in positions q_1, q_2, \dots, q_b . By separability, every row in A other than π must either have a symbol 2 in at least one of the positions p_1, p_2, \dots, p_a or a symbol 0 in at least one of the positions q_1, q_2, \dots, q_b . Let $S(p_i)$ (respectively, $S(q_i)$) be the set of rows in A with the symbol 2 (respectively, symbol 0) in position p_i (resp., q_i). Each $S(p_i)$ (resp. $S(q_i)$) must be a separable subarray of A , with separations occurring at positions other than p_i (respectively, q_i). As one 2 (one 0) is used in position p_i (resp., q_i), there are at most a many 0's and at most $b - 1$ many 2's (resp., $a - 1$ many 0's and b many 2's) that can be used to separate $S(p_i)$ (resp. $S(q_i)$). This method gives the following recursive bound on $R(n; a, b)$.

Theorem 7. For all $a, b \geq 1$ and $n \geq a + b$, $R(n; a, b) \leq 1 + aR(n - 1; a, b - 1) + bR(n - 1; a - 1, b)$.

Proof. Let A be an (n, a, b) -array of size $R(n; a, b)$. Let π be a permutation in A . Suppose that in π , the 0's are at positions p_1, \dots, p_a and the 2's are at positions q_1, \dots, q_b . Every permutation $\sigma \in A - \pi$ has a symbol 2 in at least one of the positions $p_i, 1 \leq i \leq a$ or a symbol 0 in at least one of the positions $q_j, 1 \leq j \leq b$. For every position p_i , there are at most $R(n - 1; a, b - 1)$ strings $\sigma \in A - \pi$ with $\sigma(p_i) = 2$ and, hence, a total of at most $aR(n - 1; a, b - 1)$ such strings over all p_i . For every position q_j , there are at most $R(n - 1; a - 1, b)$ strings in $\sigma \in A - \pi$ with $\sigma(q_j) = 0$ and, hence, a total of at most $bR(n - 1; a - 1, b)$ such strings over all q_j . The bound follows. \square

We can obtain an exact formula for $R(n; 1, k)$ in the next lemma and the theorem that follows.

Lemma 3. For all $k \geq 1$ and $n \geq k + 1$, $R(n; 1, k) \geq \begin{cases} n & \text{if } k + 1 \leq n \leq 2k, \\ 2k + 1 & \text{if } n \geq 2k + 1. \end{cases}$

Proof. Suppose $k + 1 \leq n \leq 2k + 1$. Let π_0 be the permutation

$$\pi_0 = (0, \underbrace{1, 1, \dots, 1}_{k \text{ times}}, \underbrace{2, 2, \dots, 2}_{n-k-1 \text{ times}}).$$

Consider permutations $\pi_0, \pi_1, \dots, \pi_{n-1}$ defined by $\pi_i(j) = \pi_0(j - i) \pmod n$, that is, π_i is obtained from π_0 by shifting elements rightward by i with wraparound.

First, we observe that the array A with rows $\pi_0, \pi_1, \dots, \pi_{n-1}$, appearing in A in order of their index, is separated. It suffices to show that row π_0 is separated from any row $\pi_i, i \geq 1$. The 0 in position 1 of π_0 separates π_0 from the 2 in column 1 of π_i for $1 \leq i \leq k$. The 2's in columns $k + 2$ through n of π_0 each separate from the 0 in the same columns for $\pi_i, k + 1 \leq i \leq n - 1$. Therefore, the permutations $\pi_i, 0 \leq i < n$ are pairwise separable and $R(n; 1, k) \geq n$.

If $n \geq 2k + 1$, then $R(n; 1, k) \geq R(2k + 1; 1, k) \geq 2k + 1$, the first inequality by monotonicity of $R(n; 1, k)$ for a fixed k (see Equation (4)) and the second by the same circular shift construction just given. \square

Theorem 8. (a) For all $k \geq 1$ and $n \geq k + 1$, $R(n; 1, k) = \begin{cases} n & \text{if } k + 1 \leq n \leq 2k, \\ 2k + 1 & \text{if } n \geq 2k + 1. \end{cases}$

(b) Suppose a separated array A has at most one 0 and at most k many 2's in each row. If A has $2k + 1$ rows, then A must have exactly one zero and k many 2's in each row.

(c) Suppose A is an $(n, 1, k)$ -array. If A has $2k + 1$ rows, then A is a $(2k + 1) \times (2k + 1)$ array also with one 0 and k many 2's in each column.

Proof. Consider first (a). In view of the lower bound in Lemma 3, it remains only to prove the corresponding upper bounds.

The upper bound $R(n; 1, k) \leq n$ follows from the fact that any two rows do not have 0 in the same position. Suppose $n \geq 2k + 1$.

Take an $(n, 1, k)$ -array with p rows. There are $\binom{p}{2} = (p - 1)p/2$ pairs of rows that have to be separated. Let Q be the total number of unordered pairs $\{0, 2\}$ with both the 0 and the 2 lying in the same column of A . Then, $\binom{p}{2} \leq Q$.

But, $Q \leq \{\text{the number of 2's that are members of such a pair (since there is only one 0 per column)}\} \leq \{\text{the total number of 2's in the array}\} = pk$.

So, we obtain $\binom{p}{2} \leq pk$. Solving for p , we obtain $p \leq 2k + 1$.

Consider now (b). Recall that no two 0's of A can be in the same column, since, otherwise, the two rows containing those 0's cannot be separated. For any column i containing a 0, let s_i be the number of 2's in column i . Since A is separated, the number of pairwise row separations in A is at least $\binom{2k+1}{2} = k(2k + 1)$. Since there is at most one 0 in each column, we have $\sum_i s_i \geq k(2k + 1)$.

Assume to the contrary that claim b) is false, so that either some row contains no 0 or some row has fewer than k many 0's. Suppose first that some row contains no 0. Then, since it has at most k many 2's, this row can be separated from most k other rows since its separation from other rows can only occur at columns containing its 2's, and each column has at most one 0. This contradicts A being separated, which requires each row to be separated from $2k$ other rows.

So we may suppose that A has $2k + 1$ many 0's, but that some row has at most $k - 1$ many 2's. Then, by the assumption in b), the total number of 2's in A is less than $(2k + 1)k$ but is also equal to $\sum_i s_i$. Then, we have $(2k + 1)k \leq \sum_i s_i < (2k + 1)k$, a contradiction.

Consider now (c). Since no two 0's of A can be in the same column, it follows that each column has exactly one 0. It also follows that A must be a $(2k + 1) \times (2k + 1)$ array.

Suppose to the contrary that some column c of A has at most $k - 1$ many 2's. Consider the row r passing through the 0 in column c . Row r must be separated from each of the $2k$ other rows of A . There are $k - 1$ separation pairs involving row r that use the 0 in column c . The remaining $k + 1$ separation pairs involving row r must use the k many 2's in row r . But each such 2 participates in only a single separation, that being with the unique 0 in its column. Hence, we cannot find $k + 1$ separations involving these 2's, a contradiction. \square

An example of a $(7, 1, 3)$ -array realizing $R(n; 1, 3) = 7$ is given in Figure 1b.

In the next lemma and theorem that follow, we use the positions technique to obtain an upper bound on $R(n : 2, k)$.

As a notation, for any subarray B of an array A , let $col(B)$ (resp. $row(B)$) be the set of columns (resp. rows) of B . Further, let B^r (resp. B^c) be the set of rows (resp. columns) of A containing entries of B . For a particular column c , $1 \leq c \leq n$, in some array A of n columns, we refer to it just by its index c . For example, for a subarray B of A , we write $c \cap B$ or $B \setminus c$ for $column(c) \cap B$ or $B \setminus column(c)$, respectively.

Now, let A be an $(n, 2, k)$ -array and let c be some column of A . Let B be the subarray consisting of all rows of A with a 0 in column c . Then, $B \setminus c$ has one 0 and k many 2's in each

row. Assume now that B has exactly $2k + 1$ rows. We then let $\text{sep}(B)$ be that $(n, 1, k)$ -array in B (guaranteed to exist by Theorem 8), which has dimensions $(2k + 1) \times (2k + 1)$.

Lemma 4. *Let A be an $(n, 2, k)$ -array and let c_1, c_2 be two distinct columns of A . Let S_1 (resp. S_2) be the subarray of A whose rows have a 0 entry in column c_1 (resp. c_2). If $|S_1| = |S_2| = 2k + 1$, then $|S_1^r \cap S_2^r| = 1$.*

Proof. Since each row of $S_1 \setminus c_1$ and $S_2 \setminus c_2$ has one 0 and k many 2's, we have $|S_i| \leq R(1, k) = 2k + 1$ by Theorem 8. By our assumption and the same theorem, we then see that $\text{sep}(S_i), i = 1, 2$, is a $(2k + 1, 1, k)$ -array with dimensions $(2k + 1) \times (2k + 1)$ and one 0 and k many 2's in each row and in each column. Note also that if $|S_1^r \cap S_2^r| \geq 2$, then any two rows in this intersection have both 0's in the same two coordinates c_1 and c_2 and, hence, cannot be separated, a contradiction to A being separated.

We are thus reduced to showing that $|S_1^r \cap S_2^r| = 0$ leads to a contradiction. Slightly abusing previous notation, in what follows, we use the term *potent symbol* to refer to either a 0 symbol or a 2 symbol in A .

Assume that $|S_1^r \cap S_2^r| = 0$. It follows that every entry in $c_1 \cap S_2$ is nonzero. So, $c_1 \cap S_2 \notin \text{col}(\text{sep}(S_2))$ since every column of $\text{sep}(S_2)$ has a 0. Since each row of $\text{sep}(S_2)$ contains one 0 and k many 2's, and since every entry of $c_2 \cap S_2$ is 0 by definition, it follows that every potent symbol in S_2 lies in $(c_2 \cap S_2) \cup \text{sep}(S_2)$. So, there remain no potent symbols of S_2 that can appear in $c_1 \cap S_2$. So, every entry in $c_1 \cap S_2$ is 1. By a symmetric argument, we also have that every potent symbol in S_1 lies in $(c_1 \cap S_1) \cup \text{sep}(S_1)$ and that every entry of $c_2 \cap S_1$ is 1. It follows that all $S_1 - S_2$ cross separations must occur in the columns contained in $\text{sep}(S_1)^c \cap \text{sep}(S_2)^c$.

The number of $S_1 - S_2$ cross separations must be at least $(2k + 1)^2$, since every row of S_1 must be separated from every row of S_2 . Now, in each column $c \in \text{sep}(S_1)^c \cap \text{sep}(S_2)^c$, there are $2k$ many $S_1 - S_2$ cross separations, obtained by pairing the 0 in $c \cap \text{sep}(S_1)$ with each of the k many 2's in $c \cap \text{sep}(S_2)$, and the same with S_1 and S_2 interchanged. Since $|\text{sep}(S_1)^c \cap \text{sep}(S_2)^c| \leq 2k + 1$, the total number of $S_1 - S_2$ cross separations is at most $2k(2k + 1) < (2k + 1)^2$, a contradiction. \square

In the theorem that follows, we abbreviate the symbols $R(n; 2, k), R(n - 1; 2, k - 1)$, and so on by $R(2, k)$ or $R(2, k - 1)$; that is, we drop the first coordinate in the R function. We take n large enough so that $R(n; 2, k)$ depends only on k (see Corollary 1). By monotonicity, the upper bound we then obtain for $R(n, 2, k)$ holds also for $R(n', 2, k)$, where $n' < n$.

Theorem 9. $R(2, k) \leq \frac{k(k + 4)(2k + 1)}{3} - 10$.

Proof. Let A be an $(n, 2, k)$ array achieving the maximum possible number of rows $R(2, k)$ for such arrays. Let π be a fixed row of A , with its two 0's in columns p_1 and p_2 and its k many 2's in columns q_1, q_2, \dots, q_k . Every row of $A \setminus \pi$, being separable from π , must have a 2 in at least one of the columns p_1 and p_2 or a 0 in at least one of the columns q_1, q_2, \dots, q_k .

Let T_1 (resp. T_2) be the subarray of $A \setminus \pi$ consisting of the rows of A with a 2 in column p_1 (resp. p_2). Note that any row of $T_1 \cup T_2$ has at most $k - 1$ many 2's outside the columns p_1, p_2 .

First, we give an upper bound for $|T_1 \cup T_2|$ as follows. Let B_1 be the set of rows in $T_1 \cup T_2$ with no 0 entry in columns p_1, p_2 . Then, $|B_1| \leq R(2, \leq k - 1) \leq R(2, k - 1)$ by Lemma 2. Let B_2 be the set of rows in $T_1 \cup T_2$ with exactly one 0 in one of the columns p_1 or p_2 . Then, by Lemma 4 and Theorem 8, we have $|B_2| \leq 2R(1, k - 1) - 1 = 4k - 3$. Finally,

no row of $T_1 \cup T_2$ can have both its 0's in columns p_1, p_2 since such a row would not be separated from π . Therefore, we have $|T_1 \cup T_2| \leq |B_1| + |B_2| \leq R(2, k - 1) + 4k - 3$.

Let $S_i, 1 \leq i \leq k$, be the subarray of A consisting of the rows of A with a 0 in column $q_i, 1 \leq i \leq k$. Note that $|S_i| \leq R(1, \leq k) \leq R(1, k) = 2k + 1$.

We now give an upper bound for $|\cup_{i=1}^k S_i^r|$. First note that for any triple of indices $1 \leq i < j < t \leq k$, we have $|S_i^r \cap S_j^r \cap S_t^r| = 0$, since any row of A contained in this triple intersection has three 0 entries, contradicting A being an $(n, 2, k)$ -array. Applying inclusion–exclusion, we thus obtain

$$|\cup_{i=1}^k S_i^r| = \sum_{1 \leq i \leq k} |S_i^r| - \sum_{1 \leq i < j \leq k} |S_i^r \cap S_j^r|. \tag{7}$$

Also note that $|S_i^r \cap S_j^r| = 0$ or 1 since any two rows of A contained in $S_i^r \cap S_j^r$ have both of their 0 entries in the same two columns q_i, q_j and, hence, cannot be separated.

We now maximize the right side of (7) over all possible collections of subarrays $S_i, 1 \leq i \leq k$ of A as defined above. Let $|S_i^r| = 2k + 1$ for $1 \leq i \leq t$, while $|S_i^r| \leq 2k$ for $t + 1 \leq i \leq k$. By Lemma 4, we have $|S_i^r \cap S_j^r| = 1$ for $1 \leq i < j \leq t$. Therefore, we obtain

$$|\cup_{i=1}^k S_i^r| \leq (2k + 1)t + (k - t)2k - \binom{t}{2} = g(t).$$

To maximize $g(t)$ on the domain $1 \leq r \leq k$, we differentiate to obtain $g'(t) = \frac{3}{2} - t$, so $t = \frac{3}{2}$ is the only critical point, and, also, $g'(1) > 0$, while $g'(2) < 0$. So, the maximum of $g(t)$ at integer values $1 \leq t \leq k$ is $\max\{g(1), g(2)\} = 2k^2 + 1$. So, we have $|\cup_{i=1}^k S_i| \leq 2k^2 + 1$.

Finally, for $k \geq 3$, we obtain the following recurrence, where the first summand “1” accounts for the fixed permutation π .

$$R(2, k) \leq 1 + |T_1 \cup T_2| + |\cup_{i=1}^k S_i^r| \leq R(2, k - 1) + 2k^2 + 4k - 1.$$

We can unravel this recurrence to obtain

$$\begin{aligned} R(2, k) &\leq R(2, 2) + 2 \sum_{i=3}^k i^2 + 4 \sum_{i=3}^k i - (k - 2) \\ &= 10 + 2 \left(\frac{k(k + 1)(2k + 1)}{6} - 1^2 - 2^2 \right) + 4 \left(\frac{k(k + 1)}{2} - 1 - 2 \right) - (k - 2) \\ &= \frac{k(k + 4)(2k + 1)}{3} - 10. \end{aligned}$$

□

We note that the upper bound on $R(n; 2, k)$ from Theorem 9, being independent of n once n is big enough, is better than the bound $R(n; 2, k) \leq \binom{n}{2}$ from Corollary 3 for n that is large relative to k , but the latter bound is stronger when $n \leq Ck^{3/2}$ for a suitable constant C . Also, the bound from Theorem 9 is stronger than the bound $R(n; 2, k) \leq \binom{2k+4}{k+2}$ from Theorem 6 for all but small k .

As examples to be used later, we mention the following.

Corollary 6. $R(2, 3) \leq 39$ and $R(2, 4) \leq 86$.

4.2. The Partition Method

In this subsection, we develop a recursive method, which we call the *partition method*, which, in some sense, generalizes the positions method of the previous subsection. In the partition method, we consider subarrays of a separable array A over $\{0, 1, 2\}$ defined by

restrictions of rows of A to a certain set of coordinates in A . In the preceding positions method, the subarrays were defined by their restriction to a single coordinate.

Let A be an (n, s, t) -array with, say, $s \leq t$. Choose a row $\pi \in A$ with s occurrences of the symbol 0 in positions p_1, \dots, p_s and t occurrences of the symbol 2 in positions q_1, \dots, q_t . For separation, all rows in A other than π must have either a symbol 2 in one of the positions p_1, \dots, p_s or a symbol 0 in one of the positions q_1, \dots, q_t . Let S be the set of strings in A with at least one 2 in the positions $\{p_1, p_2, \dots, p_s\}$ and let T be the set of strings in A with at least one 0 in the positions $\{q_1, q_2, \dots, q_t\}$. Since every string in A is separated from π , we have $A = \{\pi\} \cup S \cup T$, so $|A| \leq 1 + |S| + |T|$. In this section, we upper bound $|S|$ (and similarly $|T|$) by partitioning S into certain collections of strings, and then upper bound the sizes of each of these collections. The collections come in two types as follows. For any string $\sigma \in S$, let σ_P be the length s restriction of σ to the positions $\{p_1, p_2, \dots, p_s\}$; that is, $\sigma_P = \sigma_{p_1} \sigma_{p_2} \dots \sigma_{p_s}$. Also, let $\tau(\sigma_P)$ be the set of positions among $\{p_1, p_2, \dots, p_s\}$ at which σ_P has a 2 symbol. The two types of collections are the following.

- (1) $S_0 = \{\sigma \in S : \sigma_P \text{ has no 0 symbols}\}$.
- (2) For each nonempty subset $D \subseteq \{p_1, p_2, \dots, p_s\}$ satisfying $|D| \leq s - 1$, let $S_D = \{\sigma \in S : \sigma_P \text{ contain at least one 0 and } \tau(\sigma_P) = D\}$.

Clearly, $S = S_0 \cup \left(\bigcup_D S_D\right)$ is a partition of S , so $|S| = |S_0| + \sum_D |S_D|$. We upper bound the sizes of these sets of rows in the following lemma.

Lemma 5. *The sets S_0, S_D satisfy the following.*

- (a) *No two strings in S_0 and no two string in S_D are separable in any of the coordinates $\{p_i, 1 \leq i \leq s\}$. So, all internal separations in S_0 and in S_D occur outside the coordinates $p_i, 1 \leq i \leq s$.*
- (b) $|S_0| \leq R(n - s, s, \leq t - 1) \leq R(n - s, s, t - 1)$.
- (c) $|S_D| \leq R(n - s, \leq s - 1, t - |D|) \leq R(n - s, s - 1, t - |D|)$.

Proof. For (a), no two strings in S_0 are separable in one of the coordinates $\{p_i, 1 \leq i \leq s\}$, since neither has a 0 in those coordinates. Similarly, no two strings σ, γ in any S_D are separable in a coordinate $c \in \{p_i, 1 \leq i \leq s\}$ since we have $\sigma_c = 2$ if and only if $\gamma_c = 2$. Thus, all internal separations in S_0 or in any S_D occur in columns outside p_1, p_2, \dots, p_s . We then define the subarrays S'_0 and S'_D of A by

$$S'_0 = \{\sigma \setminus \sigma_P : \sigma \in S_0\} \text{ and } S'_D = \{\sigma \setminus \sigma_P : \sigma \in S_D\}.$$

So, S'_0 (resp. S'_D) is the set of length $n - s$ strings obtained by deleting the substring σ_P from each string $\sigma \in S_0$ (resp. $\sigma \in S_D$). Note that $|S_0| = |S'_0|$ since for any two strings $\sigma, \gamma \in S_0$, we have $\sigma \setminus \sigma_P \neq \gamma \setminus \gamma_P$ because, for some coordinate c outside p_1, p_2, \dots, p_s , we must have $\sigma_c = 2$ and $\gamma_c = 0$. This is because all internal separation in S_0 occurs outside the p_i coordinates, as observed above. Similarly, $|S_D| = |S'_D|$ for any nonempty subset $D \subseteq \{p_1, p_2, \dots, p_s\}$.

Consider now part (b). Since any $\sigma \in S_0$ has no 0's in positions $p_i, 1 \leq i \leq s$, then $\sigma \setminus \sigma_P$ is a length $n - s$ string containing s many 0's and at most $t - 1$ many 2's. Hence, $|S_0| = |S'_0| \leq R(n - s, s, \leq t - 1)$. The second inequality then follows Lemma 2.

For part (c), note that by definition for any $\gamma \in S_D$, γ_P contains at least one 0 and $|D|$ many 2's. So, $\gamma \setminus \gamma_P$ is a length $n - s$ string that has at most $s - 1$ many 0's and at most $t - |D|$ many 2's. So, we obtain $|S_D| = |S'_D| \leq R(n - s, \leq s - 1, t - |D|)$. Again, the second inequality follows Lemma 2. \square

We mention the analogue of Lemma 5 for subsets of T that correspond to S_0 and the sets S_D . For any string $\sigma \in T$, let σ_Q be the length t restriction of σ to the positions $\{q_1, q_2, \dots, q_t\}$; that is, $\sigma_Q = \sigma_{q_1} \sigma_{q_2} \dots \sigma_{q_t}$. Also, let $\tau'(\sigma_Q)$ be the set of positions among

$\{q_1, q_2, \dots, q_t\}$ at which σ_Q has a 0 symbol. In a similar way, one can define sets of rows T_0 and T_E within T as follows.

- (1) $T_2 = \{\sigma \in T : \sigma_Q \text{ has no 2 symbols}\}$.
- (2) For each nonempty subset $E \subseteq \{q_1, q_2, \dots, q_t\}$, $|E| \leq s$, let $T_E = \{\sigma \in T : \sigma_Q \text{ contain at least one 2 and } \tau'(\sigma_Q) = E\}$. The restriction $|E| \leq s$ is necessary since each row in A has at most s many 0's.

Again, we have $T = T_2 \cup \left(\bigcup_E T_E\right)$ as a partition of T , so $|T| = |T_2| + \sum_E |T_E|$. The corresponding upper bounds for $|T_2|$ and $|T_E|$ are given in the following lemma. We omit the proof as it is entirely analogous to the proof of Lemma 5.

Lemma 6. *The sets T_2 and T_E satisfy the following.*

- (a) *No two strings in T_0 and no two strings in T_E are separable in the coordinates $q_i, 1 \leq i \leq s$. So, all internal separations in T_0 and in T_E occur outside the coordinates $q_i, 1 \leq i \leq t$.*
- (b) $|T_2| \leq R(n - t, \leq s - 1, t) \leq R(n - t, s - 1, t)$
- (c) $|T_E| \leq R(n - t, \leq s - |E|, \leq t - 1) \leq R(n - t, s - |E|, t - 1)$.

We illustrate the use of the partition method for upper bounding $R(n; 3, 3)$ in the following corollary.

Corollary 7. $R(n; 3, 3) \leq 169$.

Proof. Consider an $(n, 3, 3)$ array A achieving $R(n; 3, 3)$. We find the sets S_0, S_D (with a symmetric procedure for finding the sets T_2, T_E). Then, we use Lemma 5 and other theorems to upper bound $|S_0|$ and $|S_D|$ for each $D \subset \{p_1, p_2, p_3\}$, $|D| \leq 2$. From this, we obtain a bound for S and, using Lemma 6, a symmetric bound on T . Finally, using $|A| \leq 1 + |S| + |T|$, we obtain our bound for $R(n; 3, 3)$.

Again, we take π to a row of an $(n, 3, 3)$ array A , with its three 0's in coordinates $p_1 < p_2 < p_3$ and its three 2's in coordinates $q_1 < q_2 < q_3$. We describe the sets of rows in S_0 or S_D by specifying for each row σ in such a set its length 3 restriction σ_P to p_1, p_2, p_3 . Then, we upper bound S_0 and S_D using the preceding lemmas and additional results already given. The justification for these bounds are given after the list of sets S_0 and S_D .

- 1. $S_0 = \{\sigma \in S : \sigma_P \in \{222, 221, 212, 122, 211, 121, 112\}\}, |S_0| \leq R(n - 3, 3, \leq 2) \leq 39$.
- 2. $D_1 = \{p_1, p_2\}, S_{D_1} = \{\sigma \in S : \sigma_P = \{220\}\}, |S_{D_1}| \leq R(n - 3, 2, 1) \leq 5$.
- 3. $D_2 = \{p_1, p_3\}, S_{D_2} = \{\sigma \in S : \sigma_P = \{202\}\}, |S_{D_2}| \leq R(n - 3, 2, 1) \leq 5$.
- 4. $D_3 = \{p_2, p_3\}, S_{D_3} = \{\sigma \in S : \sigma_P = \{022\}\}, |S_{D_3}| \leq R(n - 3, 2, 1) \leq 5$.
- 5. $D_4 = \{p_1\}, S_{D_4} = \{\sigma \in S : \sigma_P = \{201, 210, 200\}\}, |S_{D_4}| \leq R(n - 3, 2, 2) \leq 10$.
- 6. $D_5 = \{p_2\}, S_{D_5} = \{\sigma \in S : \sigma_P = \{021, 120, 020\}\}, |S_{D_5}| \leq R(n - 3, 2, 2) \leq 10$.
- 7. $D_6 = \{p_3\}, S_{D_6} = \{\sigma \in S : \sigma_P = \{012, 102, 002\}\}, |S_{D_6}| \leq R(n - 3, 2, 2) \leq 10$.

The bound for S_0 in item 1 comes from Theorem 9, for S_{D_i} in items 2–4 from Theorem 8, and in items 5–7 from Corollary 4. We obtain $|S| = |S_0| + \sum_D |S_D| = 84$ by symmetry $|T| = 84$ using sets T_2 and T_E , as in Lemma 6. Finally, we have $R(n; 3, 3) = |A| \leq 1 + |S| + |T| \leq 169$. \square

Note that from Lemma 1, we then have $P(d + 3, d) \leq 169$, an improvement over the previous bound $P(d + 3, d) \leq 2^6(6!) = 46,080$, cited in Theorem 1. This bound is also an improvement on the bound $P(d + 3, d) \leq \binom{12}{6} = 924$ in Corollary 5 derived from the theorem of Bollobás.

The partition technique shown in the above example is generalized in the next two theorems.

Theorem 10. For all $k \geq 3$, $R(n; k, k) \leq 1 + 2 \sum_{i=0}^{k-1} \binom{k}{i} \cdot R(n - k; k - 1, k - i)$.

Proof. Let A be an (n, k, k) array realizing $R(n; k, k)$ and let π be a row of A . As usual, we take π to have 0's in positions p_1, p_2, \dots, p_k and 2's in positions q_1, q_2, \dots, q_k . We continue with the notation S, T, S_0, S_D, T_0, T_E from the two lemmas preceding this theorem and we take $s = t = k$ in those lemmas. In particular, we have $|A| = 1 + |S| + |T|$, and we now proceed to estimate $|S|$, the estimate for $|T|$ being identical by symmetry.

By Lemma 5, we have $S_0 \leq R(n - k; k, k - 1)$.

For each subset $D \subseteq \{p_1, p_2, \dots, p_k\}$, we have by Lemma 5 that $|S_D| \leq R(n - k, k - 1, k - |D|)$. If $|D| = i$, there are $\binom{k}{i}$ such D 's. Since $|S| = |S_0| + \sum_D |S_D|$, we obtain

$$|S| \leq R(n - k; k, k - 1) + \sum_{i=1}^{k-1} \binom{k}{i} \cdot R(n - k; \leq k - 1, k - i), \text{ so} \tag{8}$$

$$|S| \leq \sum_{i=0}^{k-1} \binom{k}{i} \cdot R(n - k; \leq k - 1, k - i), \tag{9}$$

We have the same bound for T based on Lemma 6 and $|T| = |T_0| + \sum_E |T_E|$. Since $|A| = 1 + |S| + |T|$, we then obtain

$$R(n; k, k) \leq 1 + 2 \sum_{i=0}^{k-1} \binom{k}{i} \cdot R(n - k; \leq k - 1, k - i). \tag{10}$$

By Lemma 2 and Equation (4), the theorem follows. \square

Theorem 11. For all $t > s \geq 2$,

$$R(n; s, t) \leq 1 + R(n - s; s, t - 1) + R(n - t; s - 1, t) + \sum_{i=1}^{s-1} \binom{s}{i} \cdot R(n - s; s - 1, t - i) + \sum_{i=1}^s \binom{t}{i} \cdot R(n - t; s - i, t - 1).$$

Proof. We continue with the notation of Theorem 10 and the Lemmas that precede it.

Using exactly the same reasoning as in Theorem 10, we obtain

$$|S| \leq R(n - s; s, t - 1) + \sum_{i=1}^{s-1} \binom{s}{i} \cdot R(n - s; s - 1, t - i).$$

The estimate for T is very similar, except for a restriction on the sizes of sets E defining the sets T_E .

By Lemma 6, we have $T_2 \leq R(n - t, s - 1, t)$. By the same lemma, we have that for any $E \subset \{q_1, q_2, \dots, q_t\}$ with the size restriction $|E| \leq s$, we have $|T_E| \leq R(n - t, s - |E|, t - 1)$. Since there are $\binom{t}{i}, 1 \leq i \leq s$ choices for the set E , we obtain

$$|T| \leq R(n - t; s - 1, t) + \sum_{i=1}^s \binom{t}{i} \cdot R(n - t; s - i, t - 1). \tag{11}$$

Finally, the theorem follows from Equations (11) and the preceding bound for $|S|$. \square

We now calculate some values from the above recurrences.

Corollary 8. $R(n; 3, 4) \leq 605, R(n; 4, 4) \leq 3087, R(n; 3, 5) \leq 1,669, R(n; 4, 5) \leq 12,327$, and $R(n; 5, 5) \leq 69,435$.

Proof. We denote $R(n; s, t)$ by $R(s, t)$ for short (using monotonicity of $R(n; s, t)$ in n). Then, the bounds in Theorems 10 and 11 can be written as

$$\begin{aligned}
 R(k, k) &\leq 1 + 2 \sum_{i=0}^{k-1} \binom{k}{i} \cdot R(k-1, k-i) \\
 R(s, t) &\leq 1 + \sum_{i=0}^{s-1} \binom{s}{i} \cdot R(s-1, t-i) + \sum_{i=0}^s \binom{t}{i} \cdot R(s-i, t-1) \\
 &= 1 + (s+t)R(s-1, t-1) + \sum_{1 \neq i=0}^{s-1} \binom{s}{i} \cdot R(s-1, t-i) + \sum_{1 \neq i=0}^s \binom{t}{i} \cdot R(s-i, t-1).
 \end{aligned}$$

For starting values in these recurrences, we use Theorem 9 for $R(2, 3) \leq 39$ and $R(2, 4) \leq 86$, Corollary 7 for $R(3, 3) \leq 169$, Theorem 8 for $R(3, 1) = 7$, Corollary 4 for $R(2, 2) = 10$, and $R(0, k) = 1$ for all k . Now, applying the recurrences, we obtain the following values.

1. $R(3, 4) \leq 1 + 7R(2, 3) + R(2, 4) + 3R(2, 2) + R(3, 3) + 6R(1, 3) + 4R(0, 3) \leq 1 + 7 \cdot 39 + 86 + 3 \cdot 10 + 169 + 6 \cdot 7 + 4 \cdot 1 = 605.$
2. $R(4, 4) \leq 1 + 2(R(3, 4) + 4R(3, 3) + 6R(3, 2) + 4R(3, 1)) \leq 1 + 2(605 + 4 \cdot 169 + 6 \cdot 39 + 4 \cdot 7) = 3087.$
3. $R(3, 5) \leq 1 + 8R(2, 4) + R(2, 5) + 3R(2, 3) + R(3, 4) + 10R(1, 4) + 10R(0, 4) \leq 1 + 8 \cdot 86 + 158 + 3 \cdot 39 + 605 + 10 \cdot 9 + 10 = 1669.$
4. $R(4, 5) \leq 1 + 9R(3, 4) + (R(3, 5) + 6R(3, 3) + 4R(3, 2)) + (R(4, 4) + 10R(2, 4) + 10R(1, 4) + 5R(0, 4)) \leq 1 + 9 \cdot 605 + 1669 + 6 \cdot 169 + 4 \cdot 39 + 3087 + 10 \cdot 86 + 10 \cdot 9 + 5 = 12,327.$
5. $R(5, 5) \leq 1 + 2(R(4, 5) + 5R(4, 4) + 10R(4, 3) + 10R(4, 2) + 5R(4, 1)) \leq 1 + 2(12327 + 5 \cdot 3087 + 10 \cdot 605 + 10 \cdot 86 + 5 \cdot 9) = 69,435.$

□

By Lemma 1, we have $P(n, n-4) \leq 3087$, so in the notation of Theorem 1, we have $c_4 \leq 3087$. This is an improvement over that given in inequality (1), namely, $c_4 \leq 2^8(8!) = 10,321,920$. It is also an improvement on the bound $P(n, n-4) \leq \binom{16}{8} = 12,870$ derived from Corollary 5 based on the reduction from the theorem of Bollobás. The latter bound is still best though for large r .

Similarly, from the bound $R(5, 5) \leq 69,435$, we obtain $c_5 \leq 69,435$. This improves considerably the bound $c_5 \leq 2^{10}(10!)$, which is roughly 3.6×10^9 .

A rough upper bound for $R(k, k)$ obtained by applying the positions technique is $R(k, k) \leq k^{k-1} \left(\frac{e}{2}\right)^k$. Since $c_k \leq R(k, k)$ (for n large enough), this is also a considerable improvement on the bound for c_k from inequality (1). The positions and partition techniques give good bounds for $R(k, k)$ (and, hence, c_k) for moderately large k , but, still, the best such bounds so far for large k come from Corollary 5.

Author Contributions: Conceptualization, S.B., Z.M. and I.H.S.; methodology, S.B., Z.M. and I.H.S.; validation, S.B., Z.M. and I.H.S.; formal analysis, S.B., Z.M. and I.H.S.; investigation, S.B., Z.M. and I.H.S.; resources, S.B., Z.M. and I.H.S.; writing—original draft preparation, S.B., Z.M. and I.H.S.; writing—review and editing, S.B., Z.M. and I.H.S.; visualization, S.B., Z.M. and I.H.S. Each author contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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