A Saturation Problem in Meshes

Zevi Miller * Walker Yane †

January 23, 2025

Abstract

Let H and F be graphs, where we view H as the "host" graph and F as a "forbidden" graph. A spanning subgraph G of H is called *F-saturated in H* if G contains no subgraph isomorphic to F, but $G+e$ contains F for any edge $e \in E(H)-E(G)$. We let $Sat(H,F)$ be the minimum number of edges in any graph G which is F-saturated in H, where $Sat(H,F)=|E(H)|$ if H contains no copy of F as a subgraph. Most of the literature on $Sat(H,F)$ is on the case where H is the complete graph, but there has also been research on different hosts, including complete bipartite graphs, complete multipartite graphs, complete multipartite hypergraphs, and the random graph model $G(n,p)$.

Let P_m^r be the r-dimensional mesh (or grid), with vertex set $V=\{x=(x_1,x_2,\cdots,x_r): x_i$ integer, $1\le x_i\le m\}$ and edge set $E=\{xy: x,y\in V$ and $\sum_{i=1}^r |x_i-y_i|=1\}$. Let $K_{1,t}$ be the star graph on t leaves. Motivated by the classic result on $Sat(K_n,K_{1,t})$ and recent work on $Sat(G(n,p),K_{1,t})$ in the random graph model $G(n,p)$, we study $Sat(P_m^r,K_{1,t}), 2\le t\le 2r=\Delta(P_m^r)$.

We give asymptotically exact results for $Sat(P_m^2,K_{1,t}), 2\le t\le 4$. We also give upper bounds for $Sat(P_m^r,K_{1,t})$, $r\ge 3$, which are within a factor of 2 from optimal when $r=o(m)$. These two results are based on constructions, as well as lower bounds we obtain for $Sat(P_m^r,K_{1,t})$ for $r\ge 2$ and $t\ge 2$. Finally for arbitrary $r\ge 2$ we obtain asymptotically exact results for $Sat(P_m^r,K_{1,2})$, thereby showing the asymptotic behavior of the minimum size of a maximal matching in P_m^r. This result is based on a construction for the upper bound, and an edge weighting argument for the matching lower bound.

1 Introduction

1.1 Background and related work

Suppose $\mathcal{F}$ is a family of graphs. We say that a graph G is *$\mathcal{F}$-saturated* if G contains no subgraph isomorphic to an element of $\mathcal{F}$, but for any edge e in the complement of G, $G+e$ contains a subgraph isomorphic to some $F\in\mathcal{F}$. When $\mathcal{F}=F$ is a single graph F, and G satisfies the preceding conditions, then we say that G is F-saturated. The well studied extremal function $ex(n,\mathcal{F})$ is the maximum number of edges in any $\mathcal{F}$-saturated graph on n vertices. A natural dual to $ex(n,\mathcal{F})$ is the saturation function $Sat(n,\mathcal{F})$, which is the minimum number of edges in any $\mathcal{F}$-saturated graph on n vertices. We write $Sat(n,F)$ and $ex(n,F)$ for these two functions when $\mathcal{F}=F$; that is, when $\mathcal{F}$ consists of the single graph F. The best known upper bound on $Sat(n,\mathcal{F})$ for an arbitrary family $\mathcal{F}$ is given in [16].

A natural first case to study for the saturation function is $Sat(n,K_r)$, where K_r is the complete graph on r vertices. It was shown in [27] and later independently in [10] that $Sat(n,K_r)=(r-2)(n-r+2)+\binom{r-2}{2}$, and that the upper bound is uniquely realized by the join $K_{r-2}+\overline{K}_{n-r+2}$; that is, the disjoint union of K_{r-2} with a set of $n-r+2$ independent points, together with all $(r-2)(n-r+2)$ possible edges joining the set of independent points with the vertices of the K_{r-2}. At the opposite extreme in edge density from K_r for connected graphs is the star $K_{1,t}$, where there is the following result.

*Dept. of Mathematics, Miami University, Oxford, OH 45056, USA, millerz@miamioh.edu

†Dept. of Mathematics, St. Louis University High School, St. Louis, Missouri, wyane@sluh.org

Theorem 1 *[16] The values of $Sat(n, K_{1,t})$ are given as follows.*
a) *If $n \geq t + \lfloor \frac{t}{2} \rfloor$, then $Sat(n, K_{1,t}) = \binom{t-1}{2}n - \frac{1}{2}\lfloor \frac{t^2}{4} \rfloor$.*
b) *$t + 1 \leq n < t + \lfloor \frac{t}{2} \rfloor$, then $Sat(n, K_{1,t}) = \binom{t}{2} + \binom{n-t}{2}$.*

Let H be a fixed graph (where we think of H as the "host"), and F a given graph (where we think of F as the "forbidden" graph). A spanning subgraph G of H is called *F-saturated in H* (or just *F-saturated* if H is understood) if F is not a subgraph of G, but for any edge $e \in E(H) - E(G)$ the graph $G+e$ contains a subgraph of H isomorphic to F. We then let $ex(H, F)$ and $Sat(H, F)$ be the maximum and minimum (respectively) number of edges in a graph G which is F-saturated in H. If F is not a subgraph of H then we let $Sat(H, F) = ex(H, F) = |E(H)|$. Observe also that $Sat(n, F) = Sat(K_n, F)$ and $ex(n, F) = ex(K_n, F)$.

This saturation problem for general host graphs H was first mentioned in [10]. Erdos examined $Sat(H, K_3)$ in [9]. In [10] the value of $Sat(K_{r,s}, K_{m,n})$, $m \leq r$ and $n \leq s$, was conjectured, where we specify that the m-set (resp. n-set) of the $K_{m,n}$ occur in the r-set (resp. s-set) of the $K_{r,s}$. The conjecture was confirmed independently by Bollobas [3], [4], and Wessel [23], [24], where they showed that $Sat(K_{r,s}, K_{m,n}) = rs - (r - m + 1)(s - n + 1)$. Let $Sat'(K_{r,s}, K_{m,n})$ be the saturation function for the unordered version of this problem; that is, where we allow the m-set of $K_{m,n}$ to be either in the r-set or the s-set of $K_{r,s}$, and of course the n-set would be in the opposite set of $K_{r,s}$. In [21] it was conjectured that $Sat'(K_{r,r}, K_{m,n}) = (m + n - 2)r - \lfloor \left(\frac{m+n-2}{2}\right)^2 \rfloor$. In [13] it was shown that $Sat'(K_{r,r}, K_{m,n}) \geq (m + n - 2)r - (m + n - s)^2$ and that the conjecture holds in the case $m = 2$ and $n = 3$.

Another host graph that has been considered is the complete multipartite graph. The above saturation results for bipartite hosts were generalized to k-uniform multipartite hypergraph hosts by Alon [1]. In this paper he proves a result on extremal sets using methods of multilinear algebra, and from this derives results on saturation in multipartite hypergraphs. The bipartite host results were also generalized by Pikhurko in his Ph.D. thesis at Cambridge [22]. Now let K_k^n be the complete multipartite graph on k partite sets, each of size n. In [11] the values of $Sat(K_k^n, K_3)$ were determined for all $k \geq 4$ for large enough n, and also determined for $Sat(K_3^n, K_3)$ for all n.

Recently there has been work on saturation where the host is drawn from the random graph model $G(n, p)$. Here the graph $G \in G(n, p)$ has n labeled vertices, and any edge occurs with probability $p \in (0, 1)$ independent of other edges. We say that an event (i.e. a graph property) W_n in $G(n, p)$ occurs with high probability (whp) if $P(W_n) \to 1$ as $n \to \infty$. The results which follow all hold with p constant in $G(n, p)$. In [18] it was shown that whp we have $Sat(G(n, p), K_s) = (1 + o(1))nlog_{\frac{1}{1-p}}(n)$ for any fixed integer $s \geq 3$. There has also been work on $Sat(G(n, p), K_{1,t})$; that is, where the forbidden graph is the star $K_{1,t}$. Observe that $Sat(G, K_{1,2})$ is just the minimum size of a maximal matching in G. In that context it was shown in [26] that whp

$$\frac{n}{2} - log_{\frac{1}{1-p}}(np) \leq Sat(G(n, p), K_{1,2}) \leq \frac{n}{2} - log_{\frac{1}{1-p}}(\sqrt{n}).$$

In [19] it was shown that for any fixed integer $t \geq 2$ we have whp

$$Sat(G(n, p), K_{1,t}) = \frac{(t-1)n}{2} - (1 + o(1))(t - 1)log_{\frac{1}{1-p}}(n).$$

Finally the preceding result was strengthened when it was shown in [8] that $Sat(G(n, p), K_{1,t})$ is concentrated whp at two successive values.

Moving closer to "gridlike" host graphs, we mention work on saturation in the hypercube, denoting by Q_n the hypercube of dimension n. In [6] it was shown that $Sat(Q_n, Q_2) \leq (\frac{1}{4} + o(1))|E(Q_n)|$, and conjectured that this bound was best possible. In [15] this conjecture was disproved, where it was shown that for every fixed m, there exists a Q_m-saturated subgraph of Q_n with $o(|E(Q_n))$ edges. They also improved on the earlier bound on $Sat(Q_n, Q_2)$ by showing that $Sat(Q_n, Q_2) < 10 \cdot 2^n$ and asked for which m is it true that $Sat(Q_n, Q_m) = O(2^n)$. It was shown in [20] that this holds for every $m \geq 2$, specifically, that $Sat(Q_n, Q_m) \leq (1 + o(1))72m^2 2^n$.

In this paper we study saturation in the host graph P_m^r, the r-dimensional grid with coordinate entries taken from $\{1, 2, \cdots, m\}$. Here we consider $Sat(P_m^r, K_{1,t})$, where again $K_{1,t}$ is the star on t leaves.

Work on $Sat(P_m \times P_n, K_{1,t})$ for $t = 2$ has been previously done under different names. We say that a set $S \subset E(G)$ is an *edge dominating set in* G if any edge $e \in E(G)$ is either in S or is incident on some edge of S. The *edge domination number* $\gamma'(G)$ of G is the minimum size of any edge dominating set in G. An *independent* edge dominating set in G is an edge dominating set in G which forms a matching in G. It follows from a result in [2] that $\gamma'(G)$ can be realized by an independent edge dominating set in G. Clearly the graph induced by any independent edge dominating set in G of size $\gamma'(G)$ is a $K_{1,2}$-saturated subgraph of G realizing $Sat(G, K_{1,2})$. Thus $Sat(G, K_{1,2}) = \gamma'(G)$ provided G contains a copy of $K_{1,2}$. In [17] the parameter $\gamma'(G)$ was studied, and translating those results we obtain the following in our language.

Theorem 2 *[17]*
a) $\lceil \frac{mn}{3} \rceil \leq Sat(P_m \times P_n, K_{1,2}) \leq \frac{mn}{3} + \frac{n}{12} + 1$
b) $Sat(P_m \times P_n, K_{1,2}) = \lceil \frac{mn}{3} \rceil$ *when* mn *is a multiple of* 3.

Now let $m(G)$ be the minimum size of a maximal matching in G, so that $Sat(G, K_{1,2}) = m(G)$. The function $m(Q_n)$ was studied in [12], where it was proved that $m(Q_n) \geq \dfrac{2^n n}{3n-1}$ and that $\lim_{n\to\infty} \dfrac{m(Q_n)}{2^n} = \frac{1}{3}$. So again the same inequality and limit holds for $Sat(Q_n, K_{1,2})$ in place of $m(Q_n)$.

Finally we mention the dynamic survey [7] which gives a broad and detailed coverage of the area of saturation in graphs and hypergraphs. Another useful survey is given by Gould in [14], among other works on saturation by this author.

1.2 Definitions and saturation in grids

We begin with some definitions, starting with the multidimensional grid P_m^r for positive integers $r \geq 2$ and $m \geq 3$. It has vertex set $V(P_m^r) = \{x = (x_1, x_2, \cdots, x_r) : x_i \text{ an integer with } 1 \leq x_i \leq m\}$, so vertices are r-tuples x with i'th coordinate x_i an integer lying between 1 and m. The edge set is given by $E(P_m^r) = \{xy : x, y \in V(P_m^r),\ \sum_{i=1}^r |x_i - y_i| = 1\}$. So xy is an edge in P_m^r precisely when x and y disagree in exactly one coordinate, and in that coordinate they differ by 1. A straightforward induction shows that $|E(P_m^r)| = rm^r - rm^{r-1}$. We let $\partial(P_m^r)$, called the boundary of P_m^r, be the subset of $V(P_m^r)$ consisting of vertices x for which $x_i = 1$ or m for at least one i, $1 \leq i \leq r$. We call $V(P_m^r) - \partial(P_m^r)$ the *interior* of P_m^r, so this interior consists of all points $x \in P_m^r$ with $1 < x_i < m$, and hence $deg_{P_m^r}(x) = 2r$ for such x. Consider now any subgrid $P \cong P_m^d \subseteq P_m^r$, for example having vertex set $V(P) = \{x = (x_1, x_2, \cdots, x_d, a_1, a_2, \cdots, a_{r-d}) : 1 \leq x_i \leq m\}$, where $(a_1, a_2, \cdots, a_{r-d})$ is a fixed $(r-d)$-tuple, $1 \leq a_i \leq m$. In a similar way we let the boundary $\partial(P)$ of P consist of vertices $x \in P$ with $x_i = 1$ or m for at least one i, $1 \leq i \leq d$. The interior of P is then the set $V(P) - \partial(P)$, so consists of all $x \in P$ with $1 < x_i < m$, $1 \leq i \leq d$ and thus these x satisfy $deg_P(x) = 2d$. Naturally an interior point of P could be a boundary point of P_m^r, which happens if $1 < x_i < m$ for all $1 \leq i \leq d$, while $a_i = 1$ or m for at least one i, $1 \leq i \leq r - d$.

For an edge $xy \in E(P_m^r)$, we say that xy is a *k-dimensional edge* if x and y disagree by 1 in their k'th coordinates. For a graph H and subgraph $G \subset H$, we say that an edge $e \in E(H) - E(G)$ is a *nonedge of G in H*, and we let $H[\overline{G}]$ be the subgraph of H induced by the set of nonedges of G in H.

Next, given graphs G and H we let $G \times H$, called the *cartesian product* of G and H, be defined by $V(G \times H) = \{(v, w) : v \in V(G), w \in V(H)\}$ and $E(G \times H) = \{(v_1, w_1)(v_2, w_2) : v_1 = v_2$ and $w_1w_2 \in E(H)$, or $w_1 = w_2$ and $v_1v_2 \in E(G)\}$. Note that $G \times H$ can be obtained from G by replacing each vertex v of G by its own copy, call it H_v, of H, and joining two such copies H_v and $H_{v'}$ by a matching joining corresponding points in these two copies precisely when $vv' \in E(G)$. Symmetrically, $G \times H$ can also be obtained by doing the similar replacement of vertices of H by copies of G. Thus P_m^r is just the r-fold cartesian product $P_m \times P_m \times \cdots \times P_m$, where P_m is the path on m vertices. In our illustrations of $P_m^2 = P_m \times P_m$ and its subgraphs, rows are drawn from top to bottom in increasing row number. Also columns are drawn left to right in increasing column number. So point $(a, b) \in P_m^2$ is in the a'th row from the top and the b'th column from the left.

We also let P_∞^2 be the infinite two-dimensional grid having vertex set $V(P_\infty^2) = \{x = (x_1, x_2) : x_i$ an integer with $-\infty < x_i < \infty\}$, and edge set $E(P_\infty^2) = \{xy : x, y \in V(P_\infty^2),\ \sum_{i=1}^{2} |x_i - y_i| = 1\}$.

There will be additional notation introduced when needed throughout the paper. We generally use standard graph theoretic notation, as may be found for example in the texts [25] or [5].

Our results on $Sat(P_m^r, K_{1,t})$ are the following, where $2 \le t \le 2r = \Delta(P_m^r)$.

1. In dimension $r = 2$:

a) $Sat(P_m^2, K_{1,3}) = \frac{2}{3}m^2 + O(m)$.

b) $Sat(P_m^2, K_{1,4}) = \frac{6}{5}m^2 + O(m)$.

2. In dimension $r \ge 3$:

a) $Sat(P_m^r, K_{1,2}) = \frac{1}{3}m^r + O(m^{r-1})$. (This also holds for $r = 2$.)

b) For $t \ge 4$ and t even, $Sat(P_m^r, K_{1,t}) \le m^r\left(\frac{t}{2} - \frac{4}{5}\right) + Ktm^{r-1}$, where K is a constant.

c) For $t \ge 5$ and t odd, $Sat(P_m^r, K_{1,t}) \le m^r\left(\frac{t}{2} - \frac{5}{6}\right) + Ctm^{r-1}$, where C is a constant.

d) $Sat(P_m^r, K_{1,t}) \ge m^r\left(\dfrac{r(t-1)}{4r - t + 1}\right) - rm^{r-1}$. (This also holds for $r = 2$.)

The upper bounds in 1a and 1b are by construction, while the matching lower bounds in the leading coefficient follow from the general lower bound 2d. The upper bound in 2a is also by construction and the matching lower bound in the leading coefficient is obtained by an edge weighting argument. Result 2a generalizes the previously known result $Sat(P_m^2, K_{1,2}) = \frac{1}{3}m^2 + O(m)$ in dimension 2 to arbitrary dimension $r \ge 2$. The upper bound in 2b is within a factor of 2 from optimal when $r = o(m)$, also obtained by using the lower bound in 2d.

2 Bounding the saturation function of stars in the 2-dimensional grid

In this section we give constructions that give upper bounds for $Sat(P_m^2, K_{1,t})$. Later we will see that these bounds give the correct coefficient of the leading m^2 term, together with an additive $O(m)$ error term. Since $K_{1,t}$ is a subgraph of P_m^2 only for $1 \le t \le 4$, and since clearly $Sat(P_m^2, K_{1,1}) = 0$, we restrict ourselves to the cases $2 \le t \le 4$. We include the case $t = 2$ for completeness, though the results cited in Theorem 2 are more precise. Still we use the construction here for $t = 2, r = 2$ as a base for asymptotically optimal construction in higher dimensions for $t = 2$.

For each t, $2 \le t \le 4$, we define a pair of infinite spanning subgraphs $D_t(A)$ and $D_t(B)$ of P_∞^2. We illustrate the restrictions of $D_t(A)$ and $D_t(B)$ to P_5^2 in Figure 1. We let
$E(D_2(A)) = \{(i, 1 + 3k)(i, 2 + 3k) : i \text{ odd},\ k \in Z\} \cup \{(i, 2 + 3k)(i, 3 + 3k) : i \text{ even},\ k \in Z\}$ (illustrated in Figure 1a top). To avoid tedious notation we rely on Figure 1b top and 1c top to illustrate the restrictions of $D_3(A)$ and $D_4(A)$ respectively to P_5^2. The full graphs $D_3(A)$ and $D_4(A)$ are obtained by extending these illustrations to the entire P_∞^2 following the pattern shown. For example the extension of these illustrations of $D_t(A)$ to P_8^2 from P_5^2 are shown in the graphs of solid edges in Figures 2-4 (the dotted edges play a different role to be described later).

The graphs $D_t(B)$ are obtained from $D_t(A)$ by translating one row vertically down for $t = 2, 3$, and $D_4(B)$ is obtained from $D_4(A)$ by translating one column to the right. This is illustrated in the bottom row of Figure 1.

For each $2 \le t \le 4$, we let $D_{t,m}(A)$ (resp. $D_{t,m}(B)$) be the graph induced by the restriction of $D_t(A)$ (resp. $D_t(B)$) to P_m^2. That is, we have $V(D_{t,m}(A)) = V(P_m^2)$ and $E(D_{t,m}(A)) = E(D_t(A)) \cap E(P_m^2)$, with the same description for $D_{t,m}(B)$ only replacing $E(D_t(A))$ by $E(D_t(B))$. As mentioned above, letting $m = 5$, in the top row of Figure 1 we show the A versions $D_{t,m}(A)$, $2 \le t \le 4$, and in the second row of Figure 1 the B versions $D_{t,m}(B)$, $2 \le t \le 4$.

The basic properties of the graphs $D_t(A)$ and $D_t(B)$ as subgraphs of P_∞^2 are listed in the following Lemma. The properties are evident by inspecting the illustrative figures and continuing the pattern, so the proofs are omitted.

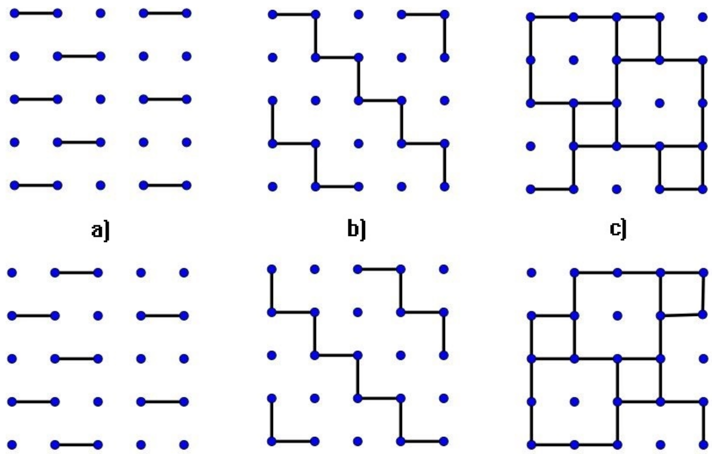

Figure 1: Portions of the graphs $D_{t,5}(A), D_{t,5}(B) \subset P_\infty^2$, $a)t = 2$, $b)t = 3$, $c)t = 4$, A version above, B version below

Lemma 3 *The graphs $D_t(A)$ and $D_t(B)$, $2 \le t \le 4$, have the following properties.*
a) *Every $v \in D_t(A)$ has $deg_{D_t}(v) = 0$ or $t-1$, and the same for $D_t(B)$.*
b) *Suppose $deg_{D_t(A)}(v) = 0$ for some $v = (a,b) \in D_t(A)$. Then each of the four points $\{(a \pm 1, b), (a, b \pm 1)\}$ in $D_t(A)$ at vertical or horizontal displacement 1 from v has degree $t-1$ in $D_t(A)$. The same holds in $D_t(B)$.*

For each t, $2 \le t \le 4$, we construct two $K_{1,t}$-saturated subgraphs, $G_t(m,2,A)$ and $G_t(m,2,B)$, of P_m^2, and $G_t(m,2,A)$ will realize our upper bound for $Sat(P_m^2, K_{1,t})$, $2 \le t \le 4$. Here $G_t(m,2,A)$ is obtained from $D_{t,m}(A)$ by adding certain edges of $E(P_m^2) - E(D_{t,m}(A))$ to $D_{t,m}(A)$, with the same description for obtaining $G_t(m,2,B)$ from $D_{t,m}(B)$.

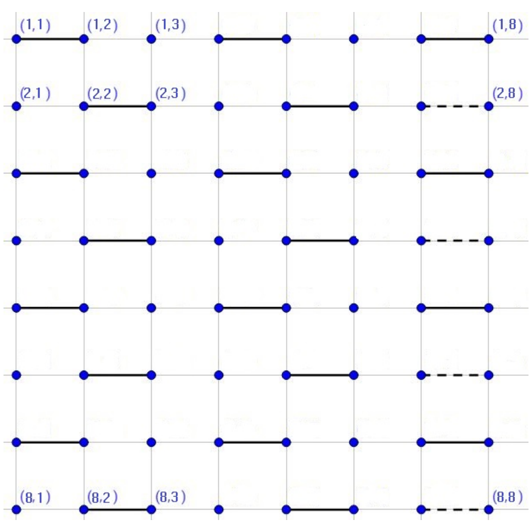

Figure 2: Graph $G_2(8,2,A)$ realizing upper bound for $Sat(P_8^2, K_{1,2})$

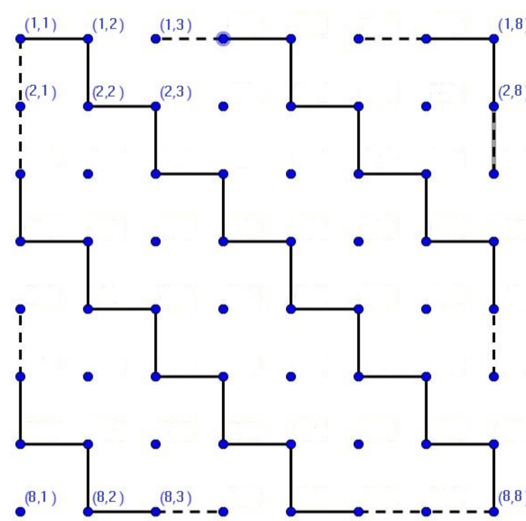

Figure 3: Graph $G_3(8,2,A)$ realizing upper bound for $Sat(P_8^2, K_{1,3})$

In fact we construct $G_t(m,2,A) = D_{t,m}(A) \cup A_{t,m}$, where $A_{t,m} \subset E(P_m^2)$ will be a set of edges touching the boundary of P_m^2 to be constructed below. Similarly we let $G_t(m,2,B) = D_{t,m}(B) \cup B_{t,m}$, where $B_{t,m} \subset E(P_m^2)$ with a similar description for $B_{t,m}$. The edges in $A_{t,8}$ are shown dotted in Figures 2-4.

We first motivate the construction of $A_{t,m}$. From Lemma 3 we know that neither $D_{t,m}(A)$ nor $D_{t,m}(B)$ contains a $K_{1,t}$, giving us a start toward a $K_{1,t}$-saturated of P_m^2. But there are edges $vw \in E(P_m^2) - D_{t,m}(A)$ (that is; nonedges of $D_{t,m}(A)$ in P_m^2) satisfying $deg_{D_{t,m}(A)}(v) < t-1$ and $deg_{D_{t,m}(A)}(w) < t-1$. A partial list of such vw can be seen in Figure 1; e.g. $(2,4)(2,5)$ in $D_{2,5}(A)$, $(5,3)(5,4)$ in $D_{3,5}(A)$, and $(4,1)(5,1)$ and $(5,3)(5,4)$ in $D_{4,5}(A)$. For such v and w, $D_{t,m}(A) + vw$ contains no $K_{1,t}$, so $D_{t,m}(A)$ is not $K_{1,t}$-saturated in P_m^2. We will greedily select a set $A_{t,m}$ of such nonedges of $D_{t,m}(A)$ in P_m^2 such that in the graph $G = D_{t,m}(A) \cup A_{t,m}$ no such v, w exist. Specifically, as long as there is a nonedge $vw \in E(P_m^2) - E(G)$ satisfying $deg_G(v) < t-1$ and $deg_G(w) < t-1$, then we add vw to $A_{t,m}$ and to the current G.

The formal construction of $G_t(m,2,A)$ and $A_{t,m}$ follows.

Construction of Graphs $G_t(m,2,A), G_t(m,2,B) \subset P_m^2, 2 \le t \le 4$

1. Initialize $A_{t,m} = \emptyset$, $G = D_{t,m}(A)$

2. While there is an edge $vw \in E(P_m^2) - E(G)$ satisfying $deg_G(v) < t-1$ and $deg_G(w) < t-1$, update
$A_{t,m} \leftarrow A_{t,m} \cup \{vw\}$
$G \leftarrow G \cup A_{t,m}$

3. When the while loop in step 2 terminates, we output $G_t(m,2,A) = G$.

The graph $G_t(m,2,B)$ is constructed by the identical procedure, by starting with $D_{t,m}(B)$ (instead of $D_{t,m}(A)$) in the initialization step 1, and building the set $B_{t,m}$ (instead of $A_{t,m}$) in the while loop of step 2. With these changes, and otherwise the identical instructions, the final step 3 gives $G_t(m,2,B) = G$. From the relation of $D_{t,m}(B)$ to $D_{t,m}(A)$, we see that apart from edges which were added greedily in step 2 (which by Observation 1 below are incident on the boundary of $D_{t,m}(B)$ by the same proof), $G_t(m,2,B)$ is obtained from $G_t(m,2,A)$ by a downward translation of one row for $t = 2,3$ or a rightward translation by one column for $t = 4$. We also note that $A_{t,m}$ and $B_{t,m}$ are not uniquely determined, but depend on the order of greedy selection of nonedges vw in step 2. We call $G_t(m,2,A)$ (resp. $G_t(m,2,B)$) a type A (resp. type B) graph, and we say the pair $G_t(m,2,A), G_t(m,2,B)$ are of opposite type.

The following observation concerning step 2 of the construction is useful.

Observation 1 *Let G be any intermediate graph in step 2 of the above procedure (including the final $G = G_t(2,A)$ or $G = G_t(2,B)$). For $x \in V(D_{t,m}(A))$, let $d(x) = deg_{D_t(A)}(x)$, and $d'(x) = deg_{D_{t,m}(A)}(x)$.*

a) *$deg_G(x) \ge d(x) \ge d'(x)$.*

b) *If x is interior in $D_{t,m}(A)$, then $d(x) = d'(x)$*

c) *If a nonedge vw of G in P_m^2 is added to G in step 2 of the construction, then either v or w is a boundary point of $D_{t,m}(A)$.*

Proof. Consider first part a), taking $x \in V(D_{t,m}(A))$. The first inequality $deg_G(x) \ge d'(x)$ holds since edges are never deleted in any iteration of step 2. The second inequality holds since $d'(x)$ does not count any edge $xy \in D_t(A)$, where $y \notin P_m^2$, though such an edge is counted in $d(x)$.

For part b), edges of $D_t(A)$ incident to x are possibly not included in $D_{t,m}(A)$ due to truncation. This happens only if x is on the boundary of $D_{t,m}(A)$. So $d(x) = d'(x)$ holds if x is interior in $D_{t,m}(A)$.

Consider part c). By Lemma 3b we have either $d(v) = t-1$ or $d(w) = t-1$. Applying parts a) and b), we see that the required condition in step 2 that $deg_G(v) < t-1$ and $deg_G(w) < t-1$ is possible only if at least one of v or w is a boundary point of $D_{t,m}(A)$. ∎

We can now summarize some simple properties of the graphs $G_t(m,2,A)$ and $G_t(m,2,B)$, $2 \le t \le 4$. We abbreviate $G_t(m,2,A)$ (resp. $G_n(m,2,B)$) by $G_t(2,A)$ (resp. $G_t(2,B)$).

Lemma 4 *The graphs $G_t(2,A)$ and $G_t(2,B)$, $2 \le t \le 4$, have the following properties.*

a) *$G_t(2,A)$ and $G_t(2,B)$ are $K_{1,t}$-saturated in P_m^2.*

b) *Suppose v and v' are corresponding interior points in $G_t(2,A)$ and $G_t(2,B)$ respectively. Then at least one of v or v' has degree $t-1$ in $G_t(2,A)$ or $G_t(2,B)$ respectively.*

Proof. Consider part a) where we restrict ourselves to $G_t(2,A)$ since the proof is identical for $G_t(2,B)$. By Lemma 3a we see that $D_{t,m}(A)$ contains no $K_{1,t}$. Each update $G \leftarrow G \cup \{vw\}$ adds a nonedge vw to G in P_m^2 for which $deg_G(v)$ and $deg_G(w)$ are both less than $t-1$ in the preceding G. Hence the updated $G \cup \{vw\}$ contains no $K_{1,t}$ if the same is true of G. Since the initial $G = D_{t,m}(A)$ contains no $K_{1,t}$, it follows by induction on the number of updates that the final output $G_t(2,A)$ contains no $K_{1,t}$, and hence $\Delta(G_t(2,A)) \le t-1$. Further, when step 2 terminates there remain no nonedges vw of $G = G_t(2,A)$ with $deg_G(v) < t-1$ and $deg_G(w) < t-1$. It follows that for every nonedge vw of the final $G = G_t(2,A)$ in P_m^2 either $deg_G(v) = t-1$ or $deg_G(w) = t-1$. So $G + vw$ contains a $K_{1,t}$, which combined with $\Delta(G_t(2,A)) \le t-1$ shows that the final $G_t(2,A)$ is $K_{1,t}$-saturated, as required.

Next consider part b). By Lemma 3a each of v and v' has degree 0 or $t-1$ as a vertex of $D_t(A)$ and $D_t(B)$ respectively. If both have degree $t-1$, there is nothing to prove. So assume by symmetry that $deg_{D_t(A)}(v) = 0$. Using Observation 1 and the relation between $D_t(A)$ and $D_t(B)$ by shifting, we see that $G_t(2,B)$ is obtained from $G_t(2,A)$ (apart from certain edges incident on the boundary) by translating one row down (for $t = 2,3$), or translating one column to the right (for $t = 4$). Therefore v' can be viewed as the point immediately above and in the same column as v in $D_t(A)$ for $t = 2,3$, or immediately to the left and in the same row as v in $D_t(A)$ for $t = 4$. So v' can be viewed as one of the four points at displacement 1 from v in $D_t(A)$. So by Lemma 3b, v' has degree $t-1$ in $D_t(B)$. Since v' is also an interior point in $G_t(2,B)$ (because it corresponds to v), it follows that v' also has degree $t-1$ in $G_t(2,B)$, as required. Clearly the same argument applies with the roles of v and v' interchanged, using a translation of one row up, or a translation of one column to the left. ■

Theorem 5 *We have the following upper bounds for $Sat(P_m^2, K_{1,t})$, $2 \le t \le 4$.*

a) *$Sat(P_m^2, K_{1,2}) \le \frac{1}{3}m^2 + O(m)$.*

b) *$Sat(P_m^2, K_{1,3}) \le \frac{2}{3}m^2 + O(m)$.*

c) *$Sat(P_m^2, K_{1,4}) \le \frac{6}{5}m^2 + O(m)$.*

Proof. We continue with the abbreviated notation $G_t(2,A)$ and $G_t(2,B)$.

By Lemma 4a we know that $G_t(2,A)$ is $K_{1,t}$-saturated in P_m^2. The upper bounds on $Sat(P_m^2, K_{1,t})$ then follow from upper bounds on $|E(G_t(2,A))|$, which we obtain as follows. The same result can be obtained by upper bounding $|E(G_t(2,B))|$ in the same way.

Consider first the case the case $t = 4$. Using Observation 1, we see that apart from roundoff error caused by possible edges of $A_{t,m}$ necessarily incident on the boundary of $G_4(2,A)$, each row of $G_4(2,A)$ can be partitioned into successive sets of 5 vertices such that 4 of these 5 vertices have degree 3, while the fifth has degree 0, as illustrated in Figure 4. So by the degree sum formula we get $|E(G_4(2,A))| \le \frac{6}{5}m^2 + O(m)$. Here the $O(m)$ term accounts for the roundoff error mentioned since there are m rows, and each endpoint of a row has at most $t-1 = O(1)$ edges of $A_{t,m}$ incident on it since $2 \le t \le 4$. The claimed bound for $|E(G_3(2,A))|$ follows from observing that, apart from roundoff at the boundary, there is a partition of each row of $G_3(2,A)$ into successive sets of 3 vertices such that 2 in each set have degree 2 while the third has degree 0 as shown in Figure 3. The bound for $|E(G_2(2,A))|$ follows from the partition of each row of $G_2(2,A)$, again apart from roundoff at boundary points, into successive sets of 3 vertices such that 2 in each set have degree 1 while the third has degree 0 as shown in Figure 2. ■

3 Upper bounds for for arbitrary t and dimension r

In this section we construct $K_{1,t}$-saturated subgraphs of P_m^r for $t \ge 2$ and $r \ge 3$. Suppose we have formed a graph $G \subset P_m^r$ with $\Delta(G) \le t-1$ as an intermediate step in this construction. We call a

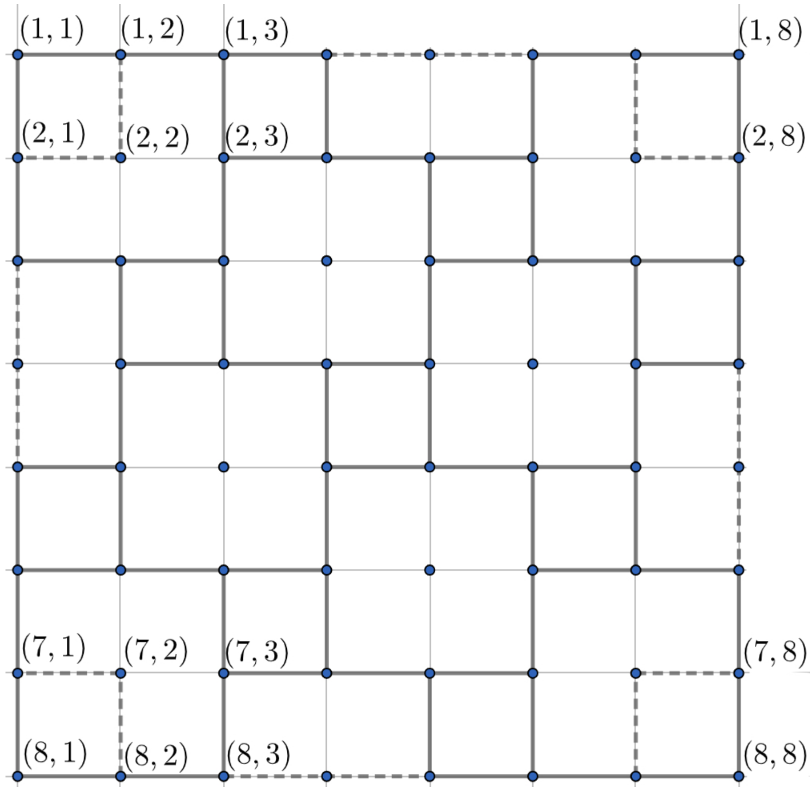

Figure 4: Graph $G_4(8, 2, A)$ realizing upper bound for $Sat(P_8^2, K_{1,4})$

nonedge vw of G in P_m^r a low degree nonedge of G if $deg_G(v) < t-1$ and $deg_G(w) < t-1$. The point here is that $G + vw$ still has degree at most $t-1$, so as long as G has any low degree nonedges, then G is not yet $K_{1,t}$-saturated. In the constructions which follow, a basic "core" graph $H \subset P_m^r$ is formed which is short of being $K_{1,t}$-saturated, after which enough low degree nonedges of H will be added to H so that the resulting graph becomes $K_{1,t}$-saturated in P_m^r.

3.1 Upper bounds for small t and arbitrary dimension $r \geq 3$

Recall the $K_{1,t}$-saturated subgraphs $G_t(m, 2, A)$ and $G_t(m, 2, B)$, $2 \leq t \leq 4$, of P_m^2 constructed in the section 2. In this subsection we construct $K_{1,t}$-saturated subgraphs of P_m^r, $2 \leq t \leq 4$ for $r \geq 3$ using these graphs.

We will need the following notation for this section.

(1) For any subgraph $G \subset P_m^r$ and $1 \leq i \leq m$ we let $G(i)$ be the subgraph of P_m^r induced by $\{v \in V(G) : v_r = i\}$, and we called $G(i)$ the *i'th layer* of G.

(2) Suppose P is a subgraph of P_m^r for some r. For any $i, 1 \leq i \leq m$, we let $P * i$ be the subgraph of P_m^{r+1} induced by the vertex set $\{(x_1, x_2, \cdots, x_r, i) : (x_1, x_2, \cdots, x_r) \in P\}$. So $P * i \cong P$.

Construction of Graphs $G_t(m, r, A), G_t(m, r, B) \subset P_m^r, 2 \leq t \leq 4, r \geq 3$

1. (Initialization with $r = 2$)
Start with the graphs $G_t(m, 2, A), G_t(m, 2, B) \subset P_m^2$ (extrapolated to any m), $2 \leq t \leq 4$, constructed in section 2, and illustrated in Figures 2-4.

2. (Inductive step) Assume inductively that we have constructed subgraphs $G_t(m, k, A)$ and $G_t(m, k, B)$ of P_m^r, $2 \leq t \leq 4$ for some $k < r$. Then for each i, $1 \leq i \leq m$, construct subgraphs $G_t(m, k+1, A)$ and $G_t(m, k+1, B)$ of P_m^{k+1} by layers as follows.

2.1 $G_t(m, k+1, A)(i) = G_t(m, k, A) * i$ for i odd, and $G_t(m, k+1, A)(i) = G_t(m, k, B) * i$ for i even,

2.2 $G_t(m, k+1, B)(i) = G_t(m, k, B) * i$ for i odd, and $G_t(m, k+1, B)(i) = G_t(m, k, A) * i$ for i even.

Now define the subgraphs $H_t(m, k+1, A)$ and $H_t(m, k+1, B)$ of P_m^{k+1} by

2.3 $H_t(m, k+1, A) = \cup_{i=1}^m G_t(m, k+1, A)(i)$ and $H_t(m, k+1, B) = \cup_{i=1}^m G_t(m, k, B)(i)$.

Set $H = H_t(m, k+1, A)$ and $H' = H_t(m, k+1, B)$.

2.4 (adding low degree nonedges of H (resp. H') in P_m^{k+1} to H (resp. H')).

While there is an edge $vw \in E(P_m^{k+1}) - E(H)$ (resp. $vw \in E(P_m^{k+1}) - E(H')$) with $deg_H(v) < t-1$ and $deg_H(w) < t-1$ (and the same degree conditions with H replaced by H')

$H \leftarrow H + vw$, $H' \leftarrow H' + vw$.

[Comment: Please see the second paragraph following this construction for motivation for step 2.4.]

2.5 (After the While loop in 2.4 terminates) If $k+1 = r$, then output $G_t(m, r, A) = H, G_t(m, r, B) = H'$ and stop.

Otherwise, $k \leftarrow k+1$, and return to step 2.

(end of construction)

We will abbreviate the graphs $G_t(m, r, A)$ (resp. $G_t(m, r, B)$) and $H_t(m, r, A)$ (resp. $H_t(m, r, B)$) by $G_t(r, A)$ (resp. $G_t(r, B)$) and $H_t(r, A)$ (resp. $H_t(r, B)$). We illustrate the inductive construction of $G_t(m, r, A)$ in Figure 5.

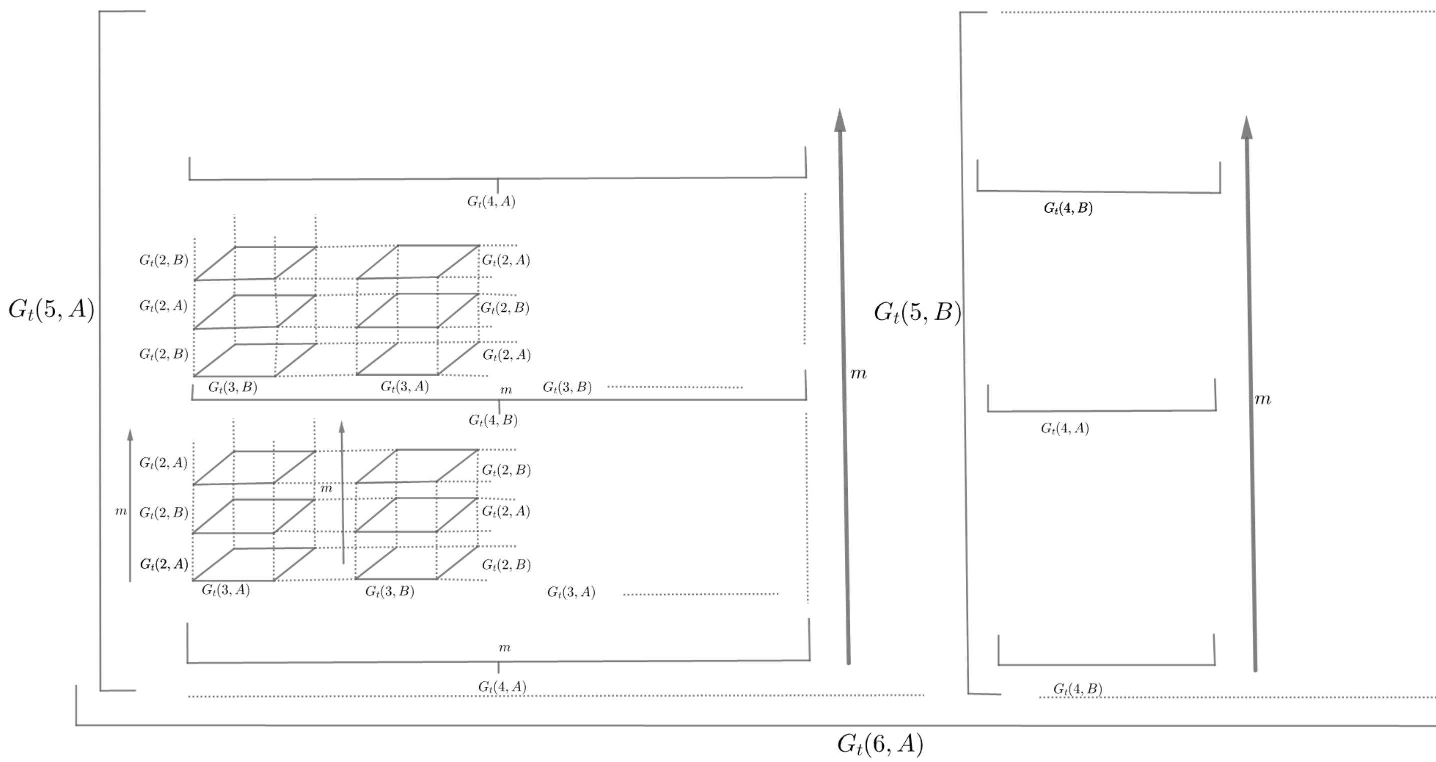

Figure 5: Inductive construction of $G_t(m, r, A)$

We motivate here step 2.4 of the preceding construction. There are instances of nonedges vw of $H_t(r+1, A)$ in P_m^r where both v and w have degree less than $t-1$ in $H_t(r+1, A)$ (and the same instances of nonedges in $H_t(r+1, B)$). Even if $\Delta(H_t(r+1, A)) \leq t-1$, so that $H_t(r+1, A)$ contains no $K_{1,t}$, still such an $H_t(r+1, A)$ is not $K_{1,t}$-saturated, since $H_t(r+1, A) + vw$ contains no $K_{1,t}$. Step 2.4 greedily adds to $H_t(r+1, A)$ enough such nonedges vw so that the resulting final graph $G_t(r+1, A) = H$, has no such low degree nonedges remaining. This graph is finally our $K_{1,t}$-saturated construction.

We begin with a basic observation.

Observation 2 *$\Delta(G_t(r, A)) \leq t-1$ and $\Delta(G_t(r, B)) \leq t-1$ for $2 \leq t \leq 4$ and $r \geq 2$.*

Proof. We prove the claim just for $G_t(r, A)$, since the proof for $G_t(r, B)$ is the same.

We proceed by induction on r. For the base case $r = 2$, we know by Lemma 4a that $G_t(2, A)$ is $K_{1,t}$-saturated, so has maximum degree at most $t - 1$.

For the inductive step assume the statement true for some $r \geq 2$ for both $G_t(r, A)$ and $G_t(r, B)$, and we prove it for $G_t(r + 1, A)$, again the proof for $G_t(r + 1, B)$ being similar and thus omitted. By construction step 2.1 we have $G_t(r + 1, A)(i) = G_t(r, A) * i \cong G_t(r, A)$ for i odd and $G_t(r + 1, A)(i) = G_t(r, B) * i \cong G_t(r, B)$ for i even. By the inductive hypothesis we know that the maximum degree in $G_t(r, A)$ and in $G_t(r, B)$ is at most $t - 1$, so it follows that $\Delta(G_t(r + 1, A)(i)) \leq t - 1$. From step 2.3 we see that $H_t(r + 1, A)$ is a disjoint union of subgraphs $G_t(r + 1, A)(i)$, so we still have $\Delta(H_t(r+1, A)) \leq t-1$. Finally consider the action of step 2.4 on $H_t(r+1, A)$. Rewriting the iterative step as $H_2 \leftarrow H_1 + vw$, we see edge vw is added to H_1 (to obtain H_2) only when $deg_{H_1}(v) < t-1$ and $deg_{H_1}(w) < t-1$. So if $\Delta(H_1) \leq t-1$, then $\Delta(H_2) \leq t-1$. Thus by induction on the number of steps in the While loop, the graph H resulting at the end of the While loop in 2.4 satisfies $\Delta(H) \leq t - 1$. Step 2.5, which sets $G_t(r + 1, A)$ to this H, then yields $\Delta(G_t(r, A)) \leq t - 1$, as required. ■

Toward showing that $G_t(r, A)$ is $K_{1,t}$-saturated, we need notation for certain 2-dimensional subgrids of P_m^r. Let $\overrightarrow{a} = (a_3, a_4, \cdots, a_r)$ be a vector of $r - 2$ fixed integers with $1 \leq a_i \leq m$ for $3 \leq i \leq r$. We let $P(\overrightarrow{a}, r)$ be set of all vertices of P_m^r having coordinates with length $r - 2$ suffix $\overrightarrow{a}$. That is, we have $P(\overrightarrow{a}, r) = \{(x_1, x_2, a_3, a_4, \cdots, a_r) : 1 \leq x_1, x_2 \leq m\}$, and we call $P(\overrightarrow{a}, r)$ a <u>2−*face* of P_m^r</u>. Now we let $P_t(\overrightarrow{a}, r, A)$ (resp. $P_t(\overrightarrow{a}, r, B)$) be the subgraph of $G_t(r, A)$ (resp. $G_t(r, B)$) induced by $P(\overrightarrow{a}, r)$. We will refer to either of the subgraphs $P_t(\overrightarrow{a}, r, A)$ and $P_t(\overrightarrow{a}, r, B)$ generically (as $\overrightarrow{a}$ varies) as a <u>2−*face* of $G_t(r, A)$</u> or <u>2−*face* of $G_t(r, B)$</u> respectively. A 2-face of either $G_t(r, A)$ or $G_t(r, B)$ is called <u>*type A*</u> (resp. <u>*type B*</u>) if it is isomorphic to $G_t(2, A)$ (resp. $G_t(r, B)$).

For a vertex $v \in P_m^r$, say with coordinates $v = (v_1, v_2, v_3, \cdots, v_r)$, we let P_v be the unique 2-face of P_m^r containing v. That is, we have $V(P_v) = \{(x_1, x_2, v_3, \cdots, v_r) : 1 \leq x_1, x_2 \leq m\}$. Notice that if $vv' \in E(P_m^r)$ is an edge of dimension $k \geq 3$, then v and v' are corresponding vertices in their respective 2-faces P_v and $P_{v'}$; that is, they have the same first two coordinates in these 2-faces and differ only in the k'th coordinate. We will say that a pair of 2-faces $P(\overrightarrow{a}, r)$ and $P(\overrightarrow{b}, r)$ are <u>*neighboring*</u> if $\sum_{i=3}^{r} |a_i - b_i| = 1$. We call such a neighboring pair of 2-faces *successive* if $|a_r - b_r| = 1$, so in that case $a_i = b_i$ for $3 \leq i \leq r - 1$.

The Lemma which follows collects certain properties of $G_t(r, A)$ and $G_t(r, B)$, including relations between their various 2-faces.

Lemma 6 *The graphs $G_t(r, A)$ and $G_t(r, B)$, $2 \leq t \leq 4$ and $r \geq 3$, have the following properties.*

a) *Let $P = P(\overrightarrow{a}, r)$ be a 2-face of P_m^r, and for $3 \leq j \leq r$ let $f_j(P) = r + \sum_{i=3}^{j} a_i$. Then $P_t(\overrightarrow{a}, r, A) \cong G_t(2, A)$ or $P_t(\overrightarrow{a}, r, A) \cong G_t(2, B)$, and the same statement holds for $P_t(\overrightarrow{a}, r, B)$. In particular, we have the following.*

a1) *If $f_r(P)$ is even, then $P_t(\overrightarrow{a}, r, A) \cong G_t(2, A)$ and $P_t(\overrightarrow{a}, r, B) \cong G_t(2, B)$. That is, $P_t(\overrightarrow{a}, r, A)$ is type A and $P_t(\overrightarrow{a}, r, B)$ is type B.*

a2) *If $f_r(P)$ is odd, then $P_t(\overrightarrow{a}, r, A) \cong G_t(2, B)$ and $P_t(\overrightarrow{a}, r, B) \cong G_t(2, A)$. That is, $P_t(\overrightarrow{a}, r, A)$ is type B and $P_t(\overrightarrow{a}, r, B)$ is type A.*

b) *Let $P = P(\overrightarrow{a}, r)$ and $P' = P(\overrightarrow{b}, r)$ be neighboring 2-faces of P_m^r. Then the pair $\{P_t(\overrightarrow{a}, r, A), P'_t(\overrightarrow{b}, r, A)\}$ are of opposite type, and the pair $\{P_t(\overrightarrow{a}, r, B), P'_t(\overrightarrow{b}, r, B)\}$ are of opposite type.*

c) *$E(G_t(r, A))$ and $E(G_t(r, B))$ contain no k-dimensional edges vv' for $k \geq 3$, where v and v' are interior points in their respective 2-faces P_v and $P_{v'}$. It follows that for any any k-dimensional edge vv' of $E(G_t(r, A))$ or $E(G_t(r, B))$, v and v' are corresponding boundary points of P_v and $P_{v'}$ respectively.*

d) *Let $vv' \in E(P_m^r) - E(G_t(r, A))$ be a nonedge of $G_t(r, A)$ in P_m^r. Then at least one of v or v' has degree $t - 1$ in $G_t(r, A)$. The same statement holds for $G_t(r, B)$.*

Proof. Throughout the proof we will abbreviate $P_t(\overrightarrow{a}, r, A)$ (resp. $P_t(\overrightarrow{a}, r, B)$) by $P_t(r, A)$ (resp. $P_t(r, B)$) since $\overrightarrow{a}$ is used generically throughout.

Consider first part a), where it suffices to prove $a1$ and $a2$. We proceed by induction on $r \geq 3$, starting with the base $r = 3$. Consider first $P_t(3, A)$, and suppose first that $f_3(P) = a_3 + 3$ is even. Then a_3 is odd, so by step 2.1 in the construction of $G_t(r, A)$ and $G_t(r, B)$, we have $G_t(3, A)(a_3) = G_t(2, A) * a_3 \cong G_t(2, A)$. Since the $\overrightarrow{a}$ vector for P is the singleton $\overrightarrow{a} = a_3$, we see that $P_t(3, A)$ is the a_3'th layer of $G_t(3, A)$. Therefore we have $P_t(3, A) = G_t(3, A)(a_3) \cong G_t(2, A)$, as required in $a1$. Now consider $a2$ for $P_t(3, A)$ where we assume $f_3(P) = a_3 + 3$ is odd. Then a_3 is even, so by step 2.1 in the construction $G_t(3, A)(a_3) = G_t(2, B) * a_3 \cong G_t(2, B)$. Again, the $\overrightarrow{a}$ vector for P being the singleton $\overrightarrow{a} = a_3$, we get $P_t(3, A) = G_t(3, A)(a_3) \cong G_t(2, B)$, as required in $a2$. The argument for proving the claims in $a1$ and $a2$ about $P_t(3, B)$ are very similar, so we omit them. This completes the base $r = 3$ of the induction.

Now suppose claims $a1$ and $a2$ are true for r, and we prove $a1$ and $a2$ for $r + 1$. An arbitrary 2-face of P_m^{r+1} can be written as $P * a_{r+1}$, where $P \subset P_m^r$ is an arbitrary 2-face assumed in a), and a_{r+1} is any integer satisfying $1 \leq a_{r+1} \leq m$. We abbreviate this 2-face by $P^+ = P * a_{r+1}$, so $V(P^+) = \{(x_1, x_2, a_3, a_4, \cdots, a_{r+1}) : 1 \leq x_1, x_2 \leq m, a_i \text{ fixed}, 3 \leq i \leq r + 1, 1 \leq a_i \leq m\}$. In what follows, recall that by definition $P_t^+(r + 1, A)$ (resp. $P_t^+(r + 1, B)$) is the subgraph $G_t(r + 1, A)$ (resp. $G_t(r + 1, B)$) induced by $V(P^+)$, and that $P_t(r, A)$ is the subgraph of $G_t(r, A)$ induced by $V(P)$. In the inductive step for $r + 1$, when proving $a1$ for example, we will show that if $f_{r+1}(P^+)$ is even, then $P_t^+(r + 1, A) \cong G_t(2, A)$ and $P_t^+(r + 1, B) \cong G_t(2, B)$. We consider the four cases defined by the independent parities of $f_{r+1}(P^+)$ and $f_r(P)$. We outline below the argument for $a1$ and $a2$ when $f_{r+1}(P^+)$ is even, for both of the cases $f_r(P)$ even and odd. The argument for $a1$ and $a2$ when $f_{r+1}(P^+)$ is odd is very similar, so we omit it.

So assume that $f_{r+1}(P^+)$ is even. We first prove $a1$ for $P_t^+(r + 1, A)$. Assume $f_r(P)$ is even. So $f_{r+1}(P^+) - f_r(P) = 1 + a_{r+1}$ is even, and hence a_{r+1} is odd. Then by step 2.1 of the construction we have $G_t(r+1, A)(a_{r+1}) = G_t(r, A) * a_{r+1}$. Therefore we have $P_t^+(r+1, A) = P_t(r, A) * a_{r+1} \cong P_t(r, A)$. Now since $f_r(P)$ is even we can apply $a1$ by induction on r to get $P_t(r, A) \cong G_t(2, A)$. It follows that $P_t^+(r + 1, A) \cong G_t(2, A)$ as required in $a1$.

Now we prove $a1$ for $P_t^+(r+1, B)$, still with $f_{r+1}(P^+)$ and $f_r(P)$ both even. As above we still have a_{r+1} odd, so by construction step 2.2 we have $G_t(r+1, B)(a_{r+1}) = G_t(r, B) * a_{r+1}$. Therefore we have $P_t^+(r + 1, B) = P_t(r, B) * a_{r+1} \cong P_t(r, B)$. Since $f_r(P)$ is even, applying $a1$ by induction on r gives $P_t(r, B) \cong G_t(2, B)$. Thus $P_t^+(r + 1, B) \cong G_t(2, B)$, as required in $a1$.

Continuing with $f_{r+1}(P^+)$ even, assume now $f_r(P)$ is odd. Since $f_{r+1}(P^+)$ even, to prove $a1$ we must still show that $P_t^+(r+1, A) \cong G_t(2, A)$ and $P_t^+(r+1, B) \cong G_t(2, B)$. Consider first $P_t^+(r+1, A)$. Here we have $f_{r+1}(P^+) - f_r(P) = 1 + a_{r+1}$ is odd, so a_{r+1} is even. By construction step 2.1 we have $G_t(r + 1, A)(a_{r+1}) = G_t(r, B) * a_{r+1}$. Therefore $P_t^+(r + 1, A) = P_t(r, B) * a_{r+1} \cong P_t(r, B)$. Now using $f_r(P)$ being odd, so that the inductive assumption for $a2$ applies, we have $P_t(r, B) \cong G_t(2, A)$. Therefore $P_t^+(r+1, A) \cong G_t(2, A)$, as required in $a1$. Next we consider $P_t^+(r+1, B)$. Then using a_{r+1} even we get from construction step 2.2 that $G_t(r + 1, B)(a_{r+}) = G_t(r, A) * a_{r+1}$. Thus $P_t^+(r + 1, B) = P_t(r, A) * a_{r+1} \cong P_t(r, A)$. Since $f_r(P)$ is odd, applying $a2$ by induction gives $P_t(r, A) \cong G_t(2, B)$. Thus $P_t^+(r + 1, B) \cong G_t(2, B)$, as required in $a1$.

Next we consider the proof of $a2$, where we assume $f_{r+1}(P^+)$ is odd. Because of the similarity with the arguments given in case $a1$, we will consider only the subcase $f_r(P)$ even since the case $f_r(P)$ odd is argued in a similar way. Here we also restrict ourselves to proving only $P_t^+(r + 1, B) \cong G_t(2, A)$, omitting the proof of $P_t^+(r + 1, A) \cong G_t(2, B)$, again because of the similarity with the case we do argue and with the arguments for $a1$. Here we have $f_{r+1}(P^+) - f_r(P) = 1 + a_{r+1}$ is odd, so a_{r+1} is even. Now $G_t(r + 1, B)(a_{r+}) = G_t(r, A) * a_{r+1}$ by construction step 2.2 since a_{r+1} is even. So $P_t^+(r + 1, B) = P_t(r, A) * a_{r+1} \cong P_t(r, A)$. Since $f_r(P)$ is even, applying $a1$ by induction on r gives $P_t(r, A) \cong G_t(2, A)$, so $P_t^+(r + 1, B) \cong G_t(2, A)$, as required in $a2$.

This completes the proof of a), and we now consider part b).

The assumption in b) implies that $|f_r(P) - f_r(P')| = 1$. Thus $f_r(P)$ and $f_r(P')$ are of opposite parity. Part b) then follows from part a).

Consider now part c), where we prove the statement for $G_t(r, A)$, the proof for $G_t(r, B)$ being essentially the same. Assume the contrary and let $vv' \in E(G_t(r, A))$ be of dimension $k \geq 3$, with

v and v' being interior points in P_v and $P_{v'}$ respectively. Then P_v and $P_{v'}$ must be distinct, with v and v' corresponding points in P_v and $P_{v'}$ respectively. Therefore v and v' disagree in exactly one of their final $r-2$ coordinates. So P_v and $P_{v'}$ are neighboring 2-faces in P^r_m. Set $Q = (P_v)_t(r, A)$ and $Q' = (P_{v'})_t(r, A)$. Then by part b), Q and Q' are of opposite type. So by Lemma 4b, either $deg_Q(v) = t-1$ or $deg_{Q'}(v') = t-1$, say $deg_Q(v) = t-1$. But $vv' \notin E(Q)$ since vv' is of dimension $k \geq 3$. Therefore $deg_{G_t(r,A)}(v) \geq 1 + deg_Q(v) = t$, a contradiction to Observation 2.

Consider now part d), where we give the proof just for $G_t(2, A)$ since the one for $G_t(2, B)$ is essentially the same. Again we work with P_v and $P_{v'}$, the 2-faces of P^r_m containing v and v' respectively, and we continue with the notation $Q = (P_v)_t(r, A)$ and $Q' = (P_{v'})_t(r, A)$. By part a) we know that Q and Q' are each either $G_t(2, A)$ or $G_t(2, B)$. If $P_v = P_{v'}$, then $Q = Q'$, so since $G_t(2, A)$ and $G_t(2, B)$ are each $K_{1,t}$-saturated by Lemma 4a, statement d) follows.

So suppose $P_v \neq P_{v'}$, and assume to the contrary that d) is false. By Observation 2 we have $\Delta(G_t(r, A)) \leq t-1$. So by the contrary assumption we have $deg_{G_t(r,A)}(v) < t-1$ and $deg_{G_t(r,A)}(v') < t-1$. But by steps 2.4 and 2.5, there should be no such nonedge vv' since the While loop was completed in the construction of of $G_t(r, A)$. ■

Corollary 7 *$G_t(r, A)$ and $G_t(r, B)$ are $K_{1,t}$-saturated for $2 \leq t \leq 4$ and $r \geq 3$.*

Proof. By Observation 2 we know that $G_t(r, A)$ contains no $K_{1,t}$ subgraph. Now take any nonedge e of $G_t(r, A)$ in P^r_m. By Lemma 6*d* at least one of the ends of e has degree $t-1$ in $G_t(r, A)$. Therefore $G_t(r, A) + e$ contains a $K_{1,t}$. The same conclusion holds for $G_t(r, B)$. ■

We can now give our upper bounds for $Sat(P^r_m, K_{1,t})$, $2 \leq t \leq 4$.

Theorem 8 *We have the following upper bounds for $Sat(P^r_m, K_{1,t})$ for $r \geq 2$ and $t \in \{2, 3, 4\}$.*
(a) $Sat(P^r_m, K_{1,2}) \leq \frac{1}{3}m^r + O(m^{r-1})$.
(b) $Sat(P^r_m, K_{1,3}) \leq \frac{2}{3}m^r + O(m^{r-1})$.
(c) $Sat(P^r_m, K_{1,4}) \leq \frac{6}{5}m^r + O(m^{r-1})$.

Proof. By Corollary 7 we know that $G_t(r, A)$ and $G_t(r, B)$ are each $K_{1,t}$-saturated for $2 \leq t \leq 4$ and any $r \geq 2$. So we have $Sat(P^r_m, K_{1,t}) \leq |E(G_t(r, A))|$ (and the same for $G_t(r, B)$). It remains to upper bound $|E(G_t(r, A))|$ and $|E(G_t(r, B))|$ to obtain the claims $(a) - (c)$. We restrict ourselves to upper bounding just $|E(G_t(r, A))|$ as the bound for $|E(G_t(r, B))|$ is identical.

By Lemma 6a,c we know that every edge of e of $G_t(r, A)$ falls into one of two types:
(1) e belongs to some 2-face $P_t(r, A) \cong G_t(2, A)$ or $G_t(2, B)$, or
(2) $e = vv'$, where v and v' are corresponding boundary points in their respective 2-faces P_v and $P_{v'}$.

The contribution to $|E(G_t(r, A))|$ from type (1) edges is $\sum |E(P)|$, where the sum is over all 2-faces $P = P_t(r, A)$ of $G_t(r, A)$. There are m^{r-2} terms in this sum, each of size $|E(G_t(2, A))|$. Upper bounds for the latter values are given in Theorem 5 implicitly, since the bounds in that theorem are bounds on $|E(G_t(2, A))|$. Applying these bounds we find that $\sum |E(P)| = m^{r-2}\big(\frac{1}{3}m^2 + O(m)\big) = \frac{1}{3}m^r + O(m^{r-1})$ for $t = 2$, and $\sum |E(P)| = m^{r-2}\big(\frac{2}{3}m^2 + O(m)\big) = \frac{2}{3}m^r + O(m^{r-1})$ for $t = 3$, and $\sum |E(P)| = m^{r-2}\big(\frac{6}{5}m^2 + O(m)\big) = \frac{6}{5}m^r + O(m^{r-1})$ for $t = 4$.

Next consider the contribution to $|E(G_t(r, A))|$ from type (2) edges. Since $G_t(r, A)$ is $K_{1,t}$-saturated by Lemma 6*d*, it follows that each vertex v of any 2-face P has degree at most $t-1 = O(1)$ in $G_t(r, A)$, since $2 \leq t \leq 4$. So this holds in particular for any boundary vertex v of P. There are $O(m)$ boundary vertices in each such P, and again m^{r-2} many 2-faces. Hence the contribution to $|E(G_t(r, A))|$ from type (2) edges is $O(m^{r-1})$, independent of t.

Finally summing the contributions from (1) and (2) for each given t we obtain the Lemma. ■

3.2 Upper bounds for $t \geq 5$ and arbitrary dimension $r \geq 3$

In this section we add edges to our constructions of the previous section to obtain upper bounds for $Sat(P_m^r, K_{1,t})$ when t is large. For this we begin by constructing $K_{1,t}$-saturated subgraphs $X_t(r)$ (resp $Y_t(r)$) of P_m^r for t even (resp. t odd). For $t = 3, 4$ we use the graphs $G_t(m, r, A)$ of the previous subsection 3.1, and will abbreviate these graphs by $G_t(r)$. We could just as well have used $G_t(m, r, B)$, but we fix ideas using the A versions. When H is a subgraph of G, we return to the notation $G[\overline{H}]$ for the graph induced in G by the set of edges $E(G) - E(H)$, so $|E(G[\overline{H}])| = |E(G)| - |E(H)|$.

We begin with t even, say $t = 2s$. To construct a $K_{1,t}$-saturated subgraph of P_m^r, we start with the cartesian product $H_s = G_4(r - s + 2) \times P_m^{s-2}$. Thus $H_s \subset P_m^r$, and H_s is obtained from $G_4(r - s + 2)$ by replacing each vertex $x \in G_4(r - s + 2)$ by a copy of P_m^{s-2} which we label $P(x)$, and each edge $xy \in G_4(r-s+2)$ by a matching M_{xy} joining each point $u \in P(x)$ to its corresponding point u' in $P(y)$. As a subgraph of P_m^r, we view $V(P(x))$ as the set of points in P_m^r whose length $r - s + 2$ suffix is the address of x in $G_4(r-s+2)$, and whose length $s-2$ prefix is the address of x in P_m^{s-2}. We then have the disjoint unions $V(P_m^r) = \cup_{x \in G_4(r-s+2)} V(P(x))$, and $E(H_s) = \big(\cup_{x \in G_4(r-s+2)} E(P(x))\big) \cup \big(\cup_{xy \in E(G_4(r-s+2))} M_{xy}\big)$.

We will see that H_s is "almost" $K_{1,t}$-saturated, in the sense that $\Delta(H_s) \leq 2s - 1$, and that most vertices have degree $2s - 1$ exactly. Still H_s fails to be $K_{1,t}$-saturated, since it has nonedges $vw \in E(P_m^r[\overline{H_s}])$ for which $deg_{H_s}(v) < 2s-1$ and $deg_{H_s}(w) < 2s-1$. Again, we greedily add such low degree nonedges to H_s until none remain, and we denote by $B_t(r)$ the set of these added nonedges. Then we let $X_t(r) = H_s \cup B_t(r)$ be our construction of a $K_{1,t}$-saturated subgraph of P_m^r.

The construction just outlined can be formalized as follows.

Construction of $K_{1,t}$-saturated Graphs $X_t(r) \subset P_m^r, 5 \leq t \leq 2r$ with $t = 2s$ even.

1. Form the graph $H_s = G_4(r - s + 2) \times P_m^{s-2}$.

2. (greedily adding low degree nonedges of H_s in P_m^r to H_s)

a) Initialize with $H' = H_s$, $B_t(r) = \emptyset$, and
$B = \{vw \in E(P_m^r[\overline{H_s}]) : deg_{H_s}(v) < 2s - 1 \text{ and } deg_{H_s}(w) < 2s - 1\}$.

b) While there is a nonedge $vw \in B$ of H' in P_m^r such that $deg_{H'}(v) < 2s-1$ and $deg_{H'}(w) < 2s-1$

b1) $B_t(r) \leftarrow B_t(r) \cup \{e\}$

b2) $H' \leftarrow H' \cup \{e\}$

b3) $B \leftarrow B - vw$

3. On completion of the while loop in step 2b, return $B_t(r)$ and $X_t(r) = H' = H_s \cup B_t(r)$.
(end of construction of $X_t(r)$).

We illustrate in Figure 6 the cartesian product $H_s = G_4(r - s + 2) \times P_m^{s-2}$, the first step in constructing $X_t(r)$. The figure shows a pair of levels in H_s which correspond to a pair of neighboring 2-faces of $G_4(r - s + 2)$, with points blown up to cubes P_m^{s-2} representing various $P(x)$'s and edges xy blown up to matchings M_{xy} joining corresponding points in the copies $P(x)$ and $P(y)$ of P_m^{s-2}. These M_{xy} are represented by either solid or dashed double lines, a distinction that will be explained below. One of the neighboring 2-faces (on top) is a blown up copy of $G_4(2, A)$, as illustrated in Figure 4 (when $m = 8$), and the neighboring 2-face (below) is a copy of $G_4(2, B)$, in accordance with Lemma 6b. The solid double lines within each level in Figure 6 are matchings M_{xy} that correspond to edges $xy \in G_4(2, A)$ or $G_4(2, B)$ that are solid in Figure 4. These edges xy come from the tesselation $D_{4,m}(A)$ at which $G_4(2, A)$ is initialized in its construction in section 2. The dashed double lines within each level of Figure 6 are matchings M_{xy} corresponding to dotted edges xy in Figure 4. These edges are low degree nonedges in P_m^2 of the current G that are added to $G_4(2, A)$ (or $G_4(2, B)$) during step 2 of the construction of $G_4(2, A)$. Also there are dashed double lines in Figure 6 joining joining the two levels, in this case joining corresponding cubes at the four corners of the two levels. These are matchings M_{xy} corresponding to low degree nonedges xy added in step 2.4 in the inductive construction of $G_4(r-s+2)$ in section 3.1.

Finally we mention that the final graph $X_t(r) = H_s \cup B_t(r)$ contains edges $vw \in B_t(r)$ that are not illustrated in Figure 6. These are added nonedges vw of H_s in P_m^r lying on the boundaries of their respective cubes $P(v)$ and $P(w)$, as described in the upcoming Corollary 10.

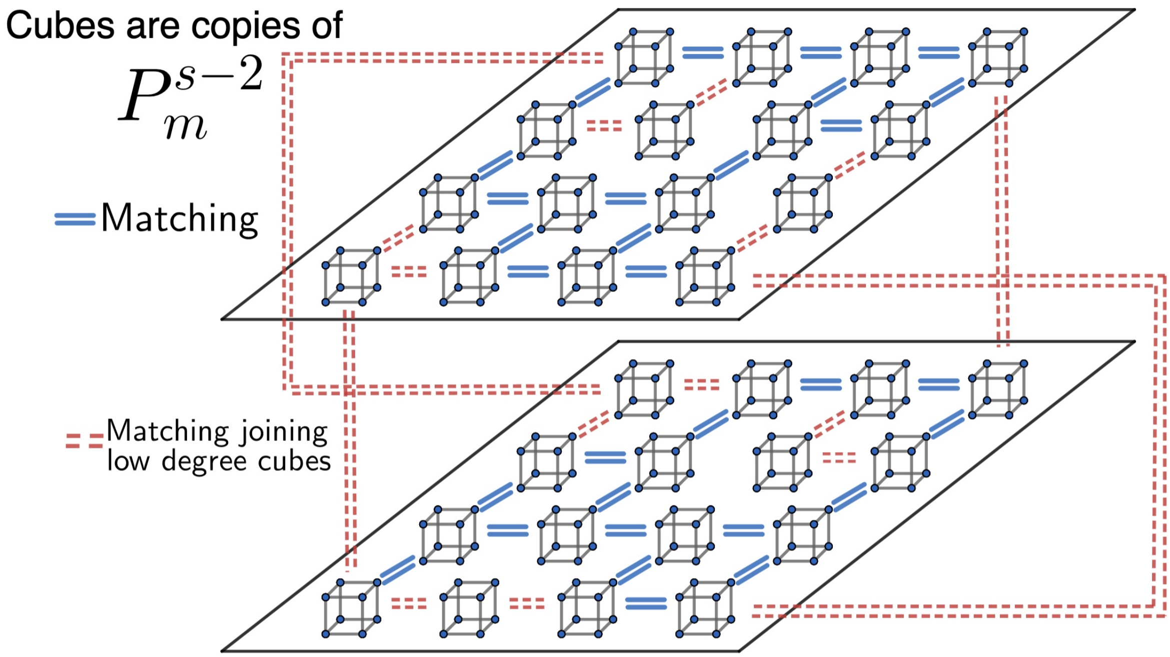

Figure 6: A portion of the graph $H_s = G_4(r-s+2) \times P_m^{s-2}$ toward building $X_t(r)$

We now analyze $X_t(r)$, leading to a proof that $X_t(r)$ is $K_{1,t}$-saturated. The next Lemma and the Corollary following will give us the tool for upper bounding $|B_t(r)|$, where we recall from step 2 of the construction of $X_t(r)$ that $B_t(r)$ is the subset of the low degree nonedges of H_s in P_m^r that were added greedily to H_s such that $X_t(r) = H_s \cup B_t(r)$.

Lemma 9 *Set $G' = G_4(r-s+2)$, and let $\mathcal{C} = \{v \in V(G') : deg_{G'}(v) = 3\}$. Also let $\beta(\mathcal{C}) = \{v \in V(H_s) : v \in \partial(P(x)), x \in \mathcal{C}\}$ and $t = 2s$.*

(a) *$\Delta(X_t(r)) \leq 2s-1$. Further this upper bound is realized at those vertices v which are internal to the $P(x)$ to which they belong with $x \in \mathcal{C}$.*

(b) *$\mathcal{C}$ is a vertex cover of the set $P_m^{r-s+2}[\overline{G'}]$ of nonedeges of G' in P_m^{r-s+2}.*

(c) *$\cup_{x\in\mathcal{C}} V(P(x))$ is a vertex cover of $P_m^r[\overline{H_s}]$.*

Proof. Consider first (a). It suffices to show that $\Delta(H_s) \leq 2s-1$. This is because by step 2 of the construction of $X_t(r)$, any edge vw in $E(X_t(r)) - E(H_s)$ is added to the current H' (which was initialized as H_s) only when $deg_{H'}(v) < 2s-1$ and $deg_{H'}(w) < 2s-1$. So $\Delta(H' \cup \{e\}) \leq 2s-1$ if $\Delta(H') \leq 2s-1$, and hence by induction on the number of steps in the while loop we get $\Delta(X_t(r)) \leq 2s-1$.

Toward showing $\Delta(H_s) \leq 2s-1$, take any $v \in V(H_s)$, say with $v \in P(x)$ for some $x \in V(G')$. Then $deg_{P(x)}(v) \leq \Delta(P(x)) = 2s-4$. Thus $deg_{H_s}(v) \leq 2s-4+3 = 2s-1$, since v has at most 3 neighbors in H_s coming from the 3 matchings M_{xy_i}, where the y_i are the at most 3 neighbors of x in $G_4(r-s+2)$. So we have $\Delta(H_s) \leq 2s-1$, and therefore by our reduction $\Delta(X_t(r)) \leq 2s-1$ as desired.

Note also that this upper bound is achieved at vertices v, say with $v \in P(x)$, satisfying $deg_{P(x)}(v) = 2s-4$, and there are three edges in H_s incident on v coming from matchings M_{xy_i}. So v is an internal vertex in $P(x)$ and $x \in \mathcal{C}$, thus proving the second statement in part (a).

For (b), first recall that G' is $K_{1,4}$-saturated in P_m^{r-s+2} by Corollary 7. Therefore any nonedge in $P_m^{r-s+2}[\overline{G'}]$ must be incident on on some vertex $x \in \mathcal{C}$, as required.

For (c), we have by step 1 of the construction of $X_t(r)$ that $V(H_s) = \cup_{x\in G'} V(P(x))$. Also every edge xy of G' becomes a matching M_{xy} in H_s joining corresponding points in $P(x)$ and $P(y)$. So take any nonedge $vw \in P_m^r[\overline{H_s}]$, say with $v \in P(y)$ and $w \in P(x)$. Then xy must be nonedge of G' in P_m^{r-s+2}. So one of x, y, say x, belongs to $\mathcal{C}$ by part (b). Hence vw is incident on $P(x)$, $x \in \mathcal{C}$, as required. ■

We can now locate the set $B_t(r)$ of nonedges H_s in P_m^r for which $X_t = H_s \cup B_t(r)$.

Corollary 10 *Take any $vw \in B_t(r)$, say with $v \in P(x)$ and $w \in P(y)$. Then at least one of x or y is in $\mathcal{C}$, and $v \in \partial P(x)$ and $w \in \partial P(y)$.*

Proof. By definition we have $B_t(r) \subset P_m^r[\overline{H_s}]$, and take any $vw \in B_t(r)$. By Lemma 9c at least one of x, y is in $\mathcal{C}$, so arbitrarily take $x \in \mathcal{C}$. If v is internal to $P(x)$, then $deg_{H_s}(v) = 2s-1$, so vw is not a low degree nonedge of H_s, contrary to the assumption $vw \in B_t(r)$. Therefore $v \in \partial P(x)$, and since w corresponds to v in $P(y)$, we also have $w \in \partial P(y)$. ■

The purpose of step 2 of the $X_t(r)$ construction is to find a set of edges $B_t(r) \subset B$ (with B defined in step 2a in constructing $X_t(r)$) such that in the final graph $X_t(r) = H_s \cup B_t(r)$ there exist no low degree nonedges $vw \in P_m^r[\overline{H}']$; that is, nonedges vw with $deg_{H'}(v) < 2s-1$ and $deg_{H'}(w) < 2s-1$. We build $B_t(r)$ greedily, maintaining the maximum degree condition $\Delta(H') \le 2s-1$ for each intermediate subgraph H' along the way; that is, where $H' = H_s \cup B_t(r)$, and where $B_t(r)$ is the set of edges that has been added to H_s so far. The final graph $X_t(r)$ will be $K_{1,t}$–saturated in P_m^r, as the next Lemma shows.

Lemma 11 *The graph $X_t(r)$ is $K_{1,t}$-saturated in P_m^r for $t = 2s$ even.*

Proof. By Lemma 9a we have $\Delta(X_t(r)) \le 2s-1$, so $X_t(r)$ contains no $K_{1,t}$. Now take any $e = vw \in E(P_m^r) - E(X_t(r))$. Since the while loop in step 2b was completed, it must be that at least one of v or w of e, say v, satisfies $deg_{X_t(r)}(v) = 2s-1$. Thus $X_t(r) + e$ contains a $K_{1,t}$. So $X_t(r)$ is $K_{1,t}$-saturated. ■

Now suppose $t = 2s-1$ is odd, and we construct in a similar way a $K_{1,t}$-saturated subgraph $Y_t(r)$ of P_m^r. The difference is in the first step, where we use $G_3(r-s+2)$ instead of $G_4(r-s+2)$ in the cartesian product. That is, we let $L_s = G_3(r-s+2) \times P_m^{s-2}$. Note that L_s is analogous to H_s for the case t even, and L_s obtained from $G_3(r-s+2)$ by replacing each vertex $x \in G_3(r-s+2)$ by a copy of P_m^{s-2} which we continue to label $P(x)$, and each edge $xy \in G_3(r-s+2)$ by a matching M_{xy} joining each point $u \in P(x)$ to its corresponding point u' in $P(y)$. As a subgraph of P_m^r, we view $V(P(x))$ as the set of points in P_m^r whose length $r-s+2$ suffix is the address of x in $G_3(r-s+2)$, and whose length $s-2$ prefix varies over all points in P_m^{s-2}. We then have the union $V(P_m^r) = \cup_{x\in G_3(r-s+2)} V(P(x))$, and $E(L_s) = \left(\cup_{x\in G_3(r-s+2)} E(P(x))\right) \cup \left(\cup_{xy\in E(G_3(r-s+2)} M_{xy}\right)$.

For completeness, we describe the construction of $Y_t(r)$ explicitly, though the similarity with the construction of $X_t(r)$ is obvious.

Construction of $K_{1,t}$-saturated Graphs $Y_t(r) \subset P_m^r, 5 \le t \le 2r$ with $t = 2s-1$ odd.

1. Form the graph $L_s = G_3(r-s+2) \times P_m^{s-2}$.

2. (greedily adding nonedges of L_s in P_m^r to L_s)

- **a)** Initialize with $L' = L_s$, $C_t(r) = \emptyset$, and
  $C = \{vw \in E(P_m^r[\overline{L_s}]) : deg_{L_s}(v) < 2s-2 \text{ and } deg_{L_s}(w) < 2s-2\}$.

- **b) While** there is a nonedge $vw \in C$ such that $deg_{L'}(v) < 2s-2$ and $deg_{L'}(w) < 2s-2$

b1) $C_t(r) \leftarrow C_t(r) \cup \{e\}$

b2) $L' \leftarrow L' \cup \{e\}$

b3) $C \leftarrow C - vw$

3. On completion of the while loop in step 2b, Return $C_t(r)$ and $Y_t(r) = L' = L_s \cup C_t(r)$.

(end of construction of $Y_t(r)$).

We now analyze $Y_t(r)$ along lines of the analysis of $X_t(r)$. Set $G_3 = G_3(m, r-s+2)$.

The proof that $\Delta(Y_t(r)) \leq 2s-2$ is reducible to showing that $\Delta(L_s) \leq 2s-2$, again since step 2 only adds edges vw of $E(P_m^r) - E(L_s)$ to the current L' when $deg_{L'}(v) < 2s-2$ and $deg_{L'}(w) < 2s-2$.

So consider $\Delta(L_s)$. Take a vertex $v \in L_s$, say with $v \in P(x)$ for some $x \in V(G_3)$. If v is interior to $P(x)$, then $deg_{P(x)}(v) = 2s-4$, so $deg_{L_s}(v) = 2s-4+2 = 2s-2$, where the additive 2 comes from two matchings M_{xy_i}, where the y_i are the at most 2 neighbors of x in G_3. If v is a boundary point of $P(x)$, then $deg_{P(x)}(v) \leq 2s-5$, leading to $deg_{L_s}(v) < 2s-2$ by the same reasoning using the matchings M_{xy_i}. So we get $\Delta(Y_t(r)) \leq 2s-2$.

Again, the purpose of step 2 of the $Y_t(r)$ construction is to find a set of nonedges $C_t(r) \subset C$ such that in the graph $L' = L_s \cup C_t(r)$ there exist no nonedges $vw \in P_m^r[\overline{L'}]$ with $deg_{L'}(v) < 2s-2$ and $deg_{L'}(w) < 2s-2$. The paragraph above shows that such a pair v, w would be corresponding boundary points in their respective $P(x)$'s; that is, $v \in \partial P(x)$ and $w \in \partial P(y)$ with $xy \in E(G_3)$ We then let $Y_t(r) = L'$, and it will follow that $Y_t(r)$ is $K_{1,t}$–saturated. We build $C_t(r)$ greedily, maintaining the maximum degree condition $\Delta(L') \leq 2s-2$ for each intermediate subgraph L' along the way; that is, where $L' = H_s \cup C_t(r)$, and where $C_t(r)$ is the set of edges that have been added to L_s so far.

Lemma 12 *The graph $Y_t(r)$ is $K_{1,t}$-saturated in P_m^r, for $t = 2s-1$ odd.*

Proof. We have already shown that $\Delta(Y_t(r)) \leq 2s-2$ in the discussion preceding this Lemma. Now take any $e = vw \in E(P_m^r) - E(Y_t(r))$, and consider $L' = Y_t(r)$ when the algorithm stops. Since the while loop in step $2b$ was completed, it must be that at least one of v or w, say v, satisfies $deg_{Y_t(r)}(v) = 2s-2$. Thus $Y_t(r) + e$ contains a $K_{1,2s-1}$. So $Y_t(r)$ is $K_{1,2s-1}$-saturated. ∎

The graphs $X_t(r)$ and $Y_r(r)$ now yield the upper bounds for $Sat(P_m^r, K_{1,t})$ in the following theorem.

Theorem 13 *Let $t \geq 5$. As m grows, we have the following.*

a) *For t even, $Sat(P_m^r, K_{1,t}) \leq m^r(\frac{t}{2} - \frac{4}{5}) + m^{r-1}\big(\frac{3t}{10}(1+o(1))\big)$.*

b) *For t odd, $Sat(P_m^r, K_{1,t}) \leq m^r(\frac{t}{2} - \frac{5}{6}) + m^{r-1}\big(\frac{t}{6}(1+o(1))\big)$.*

Proof. Suppose first that t is even, say with $t = 2s$. By Lemma 11 we know that $X_t(r)$ is $K_{1,t}$-saturated. Thus $Sat(P_m^r, K_{1,t}) \leq |E(X_t(r))|$, so we now focus on upper bounding $|E(X_t(r))|$.

We have $|E(X_t(r))| = |E(H_s)| + |B_t(r)|$. Consider first $|E(H_s)|$, and set $G_4 = G_4(m, r-s+2)$. By construction of $X_t(r)$, every edge of H_s belongs either to some $P(x) \cong P_m^{s-2}$, $x \in V(G_4)$, or to some matching M_{xy}, $xy \in E(G_4)$. We now estimate the contribution to $|E(H_s)|$ coming from each of these two categories of edges separately.

Now H_s contains $|V(G_4)| = m^{r-s+2}$ copies of P_m^{s-2} (the $P(x)$'s), each having $(s-2)m^{s-2} - (s-2)m^{s-3}$ edges. So the number of edges contained in $\cup_{x \in G_4} P(x)$ is then $(s-2)m^r - (s-2)m^{r-1}$. Next we estimate the number of edges in all matchings M_{xy}, $xy \in E(G_4)$. By Theorem 8c, G_4 has $\frac{6}{5}m^{r-s+2} + O(m^{r-s+1})$ edges, each of which gets blown up in H_s to a matching M_{xy} of size m^{s-2}. So there are $\big(\frac{6}{5}m^{r-s+2} + O(m^{r-s+1})\big)m^{s-2} = \frac{6}{5}m^r + O(m^{r-1})$ edges in $\cup_{xy \in E(G_4)} M_{xy}$. Therefore adding the contribution to $|E(H_s)|$ from these two kinds of edges we get $|E(H_s)| = (s - \frac{4}{5})m^r - (s+O(1))m^{r-1}$.

Next consider $|B_t(r)|$. Recall from Lemma 9 the notation $\mathcal{C} = \{v \in V(G_4) : deg_{G_4}(v) = 3\}$. By Lemma 9c we know that $\cup_{x \in \mathcal{C}} V(P(x))$ is a vertex cover of $P_m^r[\overline{H}_s]$. Also by Corollary 10, if $vw \in B_t(r)$, say with $v \in P(x)$ and $w \in P(y)$, then either $x \in \mathcal{C}$ or $y \in \mathcal{C}$, while $v \in \partial P(x)$ and $w \in \partial P(y)$. So for

$x \in \mathcal{C}$, let $B(P(x))$ be the number of edges in $B_t(r)$ incident on $\partial P(x)$, and let $E_1 = \sum_{x\in\mathcal{C}} B(P(x))$. Then by the preceding discussion we have generously $|B_t(r)| \le E_1$.

To estimate E_1, we have just recalled that any e counted in E_1 must be incident on some vertex v satisfying $v \in \partial P(x)$ with $x \in \mathcal{C}$. Such a v satisfies $deg_{P(x)}(v) = 2s-4-j$ for some $1 \le j \le s-2$. Since $deg_{G_4}(x) = 3$, we have $deg_{H_s}(v) = 2s-1-j$ because of the three matching edges incident on v coming from M_{xy_i}, where y_i are the 3 neighbors of x in G_4. Since $\Delta(X_t(r)) \le 2s-1$, we have that v is incident on at most j edges counted in E_1 (in fact counted in $B(P(x))$). Also since $deg_{P(x)}(v) = 2s-4-j$ and $\Delta(P_m^{s-2}) = 2s-4$, among the first $s-2$ coordinates of v as a point of P_m^{s-2}, there are j "extreme" coordinate values; that is, values that are 1 or m.

We can then count the number of such v, and from this obtain our upper bound on E_1. The number j of coordinates of v which are extreme, taken from among the $s-2$ coordinates determining $P(x)$, can be selected in $\binom{s-2}{j}$ ways. Each such coordinate has two possible values (1 or m), so there are 2^j possible sets of values in these j extreme coordinate positions. There are also $(m-2)^{s-2-j}$ possible values in the remaining $s-2-j$ coordinate positions having nonextreme values for such a v. Thus there are $2^j\binom{s-2}{j}(m-2)^{s-2-j}$ vertices $v \in P(x)$ with $deg_{P(x)}(v) = 2s-4-j$ for any $1 \le j \le s-2$ in a given $P(x)$, $x \in \mathcal{C}$. Hence we have

$$B(P(x)) \le \sum_{j=1}^{s-2} 2^j\binom{s-2}{j}(m-2)^{s-2-j}j = (m-2)^{s-2}\sum_{j=1}^{s-2}\left(\tfrac{2}{m-2}\right)^j\binom{s-2}{j}j =: g(s,m).$$

and $|E_1| \le |\mathcal{C}|g(s,m)$.

To evaluate $g(s,m)$, start with the identity $(1+x)^k - 1 = \sum_{j=1}^k \binom{k}{j}x^j$. Differentiating and then multiplying by x we get $kx(1+x)^{k-1} = \sum_{j=1}^k \binom{k}{j}x^j j$. Setting $x = \frac{2}{m-2}$ and $k = s-2$, we obtain $\sum_{j=1}^{s-2}\left(\frac{2}{m-2}\right)^j\binom{s-2}{j}j = \frac{2(s-2)}{m-2}\left(1+\frac{2}{m-2}\right)^{s-3}$, and so

$$g(s,m) = (m-2)^{s-3}2(s-2)\left(1+\tfrac{2}{m-2}\right)^{s-3} = 2(s-2)m^{s-3}.$$

Next we claim that $|\mathcal{C}| = (\frac{4}{5}+o(1))m^{r-s+2}$ as m grows. For this, let $\mathcal{C}_P$ be the restriction of $\mathcal{C}$ to any 2-face $P = P_4(r-s+2, A)$ of $G_4(r-s+2)$. Such a P is just a copy of $G_4(2,A)$ or $G_4(2,B)$ by Lemma 6a. On examining $G_4(2,A)$ (or similarly $G_4(2,B)$) in Figure 4 we see that, possibly apart from vertices on the boundary or adjacent to the boundary of P, every row of P can be partitioned into successive groups of 5 vertices, 4 of which have degree 3 in $G_4(2,A)$ (or B) and hence by construction in $G_4(r-s+2)$. These degree 3 vertices belong to $\mathcal{C}_P$, so $|\mathcal{C}_P| = \frac{4}{5}m^2 + O(m) = (\frac{4}{5}+o(1))m^2$. Since $V(G_4(r-s+2))$ is a disjoint union of m^{r-s} many such 2-faces P of $G_4(r-s+2)$, it follows that $|\mathcal{C}| = (\frac{4}{5}+o(1))m^{r-s+2}$ as m grows. So finally we obtain

$$|B_t(r)| \le E_1 \le |\mathcal{C}|g(s,m) \le (\tfrac{8}{5}+o(1))(s-2)m^{r-1}.$$

Combining our estimates for $|E(H_s)|$ and $|B_t(r)|$, we get $Sat(P_m^r, K_{1,t}) \le |E(X_t(r)|$
$= |E(H_s)| + |B_t(r)| \le m^r(s-\frac{4}{5}) - m^{r-1}(s+O(1)) + (\frac{8}{5}+o(1))(s-2)m^{r-1}$
$= m^r(s-\frac{4}{5}) + m^{r-1}\left(\frac{3}{5}s(1+o(1))\right)$. Substituting $t = 2s$, we get

$$Sat(P_m^r, K_{1,t}) \le m^r(\tfrac{t}{2}-\tfrac{4}{5}) + m^{r-1}\left(\tfrac{3t}{10}(1+o(1))\right), \text{ as required.}$$

Consider next part b), where $t = 2s-1$ is odd. By Lemma 12 we know that $Y_t(r)$ is $K_{1,t}$-saturated. So it remains only to estimate $|E(Y_t(r))|$, which serves as the required upper bound in part b). The analysis is very similar to that just given for $|E(X_t(r))|$, so we just outline it here omitting some details.

We have $|E(Y_t(r))| = |E(L_s)| + |C_t(r)|$. To estimate $|E(L_s)|$, note that L_s contains m^{r-s+2} copies of P_m^{s-2} (the $P(x)$'s), each having $(s-2)m^{s-2} - (s-2)m^{s-3}$ edges. By the construction in Theorem 8b, $G_3 = G_3(m, r-s+2)$ has $\frac{2}{3}m^{r-s+2} + O(m^{r-s+1})$ edges, each of which gets blown up in L_s to a matching M_{xy} of size m^{s-2} joining two such $P(x)'s$. So we get
$|E(L_s)| = \left(\frac{2}{3}m^{r-s+2} + O(m^{r-s+1})\right)m^{s-2} + ((s-2)m^{s-2} - (s-2)m^{s-3})m^{r-s+2}$
$= (s-\frac{4}{3})m^r - (s+O(1))m^{r-1}$.

Next we estimate $|C_t(r)|$. We let $C(P(x))$ be the number of edges of $C_t(r)$ incident on $P(x)$, and thus on $\partial P(x)$. Since G_3 is $K_{1,3}$-saturated in P_m^{r-s+2}, the set $\mathcal{D} = \{v \in V(G_3) : deg_{G_3}(v) = 2\}$ is a vertex cover of $P_m^{r-s+2}[\overline{G}_3]$. Therefore by our construction of L_s, the set $X_{\mathcal{D}} = \cup_{x\in\mathcal{D}} V(P(x))$ is a vertex cover of $P_m^r[\overline{L}_s]$. So $|C_t(r)| \le \sum_{x\in\mathcal{D}} C(P(x))$.

To estimate $C(P(x))$, consider $v \in \partial P(x)$. We have $deg_{P(x)}(v) = 2s - 4 - j$, so then $deg_{L_s}(v) = 2s - 2 - j$ because $deg_{G_3}(x) = 2$. Since $\Delta(Y_t(r)) \le 2s - 2$ such a v is incident on at most j edges of $C_t(r)$. As in the analysis of part a), we then have $C(P(x)) \le g(s,m) = 2(s-2)m^{s-3}$ for any $x \in \mathcal{D}$. By an argument similar to that given for the estimate of $\mathcal{C}$, we get $|\mathcal{D}| = (\frac{2}{3} + o(1))m^{r-s+2}$, so

$$|C_t(r)| \le (\tfrac{2}{3} + o(1))m^{r-s+2}g(s,m) = (\tfrac{4}{3} + o(1))(s-2)m^{r-1}.$$

Altogether then for $t = 2s - 1$ we have $Sat(P_m^r, K_{1,t}) \le |E(Y_t(r))| = |E(L_s)| + |C_t(r)|$
$\le (s - \frac{4}{3})m^r - (s + O(1))m^{r-1} + (\frac{4}{3} + o(1))(s-2)m^{r-1}$
$= (s - \frac{4}{3})m^r + \big(\frac{s}{3}(1 + o(1))\big)m^{r-1}$. Substituting $s = \frac{t+1}{2}$, we obtain

$$Sat(P_m^r, K_{1,t}) \le m^r(\tfrac{t}{2} - \tfrac{5}{6}) + m^{r-1}\big(\tfrac{t}{6}(1 + o(1))\big). \qquad \blacksquare$$

4 Lower bounds for $t \ge 2$ and arbitrary dimension r

4.1 The case $t = 2$

A simple lower bound for $Sat(H, K_{1,t})$ for an arbitrary host graph H, mentioned also in [19], is the following. Let $\alpha_k(H)$ be the maximum size of a subset $S \subset V(H)$ which induces a subgraph of H of maximum degree at most k.

Lemma 14 *Let $n = |V(H)|$. Then $Sat(H, K_{1,t}) \ge \frac{1}{2}(t-1)(n - \alpha_{t-2}(H))$.*

Proof. Let G be a $K_{1,t}$-saturated subgraph of H, and let A be the set of vertices in G of degree at most $t-2$ in G. We have $deg_G(x) = t - 1$ for every vertex $x \in V(H) - A$. There can be no edge $e \in E(H) - E(G)$ both of whose ends lie in A, since then $G + e$ would contain no $K_{1,t}$, and hence G would not be $K_{1,t}$-saturated. Therefore $[A]_H = [A]_G$, so $|A| \le \alpha_{t-2}(H)$. It follows that $|E(G)| \ge \sum_{x \in V(H)-A} deg_G(x) \ge \frac{1}{2}(t-1)(n - \alpha_{t-2}(H))$. $\blacksquare$

Note that $\alpha_0(H)$ is the independence number of H. Applying the Lemma to our host graph P_m^r, setting $t = 2$, and observing that $\alpha_0(P_m^r) = \lceil \frac{m^r}{2} \rceil$, we obtain $Sat(P_m^r, K_{1,2}) \ge \frac{m^r}{4}$ as a lower bound. We improve on that bound in this subsection by showing that $Sat(P_m^r, K_{1,2}) \ge \frac{m^r}{3}(1 + o(1))$ as m grows. The leading coefficient here matches that of our upper bound construction to yield $Sat(P_m^r, K_{1,2}) = \frac{m^r}{3}(1 + o(1))$ as m grows.

Our lower bound for $Sat(P_m^r, K_{1,2})$ is based on an edge weighting argument. The first step for this is a lemma counting edge-nonedge incidences, for which we need the following definitions. Let G_r be a $K_{1,2}$-saturated subgraph of P_m^r. Thus G_r is a maximal matching in P_m^r. For any point $z = (z_1, z_2, \cdots, z_r) \in P_m^r$, and any $1 \le s \le r$, let $z(s+) = (z_1, z_2, \cdots, z_{s-1}, z_s + 1, z_{s+1}, \cdots, z_r)$ and $z(s-) = (z_1, z_2, \cdots, z_{s-1}, z_s - 1, z_{s+1}, \cdots, z_r)$, provided $z_s \le m - 1$ for $z(s+)$ and $z_s \ge 2$ for $z(s-)$. So $z(s+)$ (rep. $z(s-)$) is obtained from z by adding (resp. subtracting) 1 to the s'th coordinate, and leaving all other coordinates fixed. For any nonedge $e' \in E(P_m^r) - E(G_r)$ of G_r in P_m^r and $e \in G_r$ we write $e' \sim e$ to indicate that e' is incident to e; that is, e and e' share an endpoint. We define the following.

(1) For $e \in E(G_r)$ we let $\mathbf{g}(\mathbf{e}) = |\{\text{nonedges } e' \text{ of } G_r : e' \sim e\}|$.

(2) Take $e \in E(G_r)$ and suppose e is of dimension d, $1 \le d \le r$. Then let
$\mathbf{f}(\mathbf{e}) = |\{$nonedges e' of G_r in P_m^r: e' is incident to two edges of G_r one of which is e, and e' has dimension not equal to $d\}|$.

(3) For any vertex $v \in P_m^r$ let $\mathbf{b}(\mathbf{v})$ be the number of coordinates of v with the extreme values 1 or m.

(4) For $e = xy \in E(G_r)$, let $\mathbf{c}(\mathbf{e}) = \max\{b(x), b(y)\}$.

Lemma 15 *Let G_r be a $K_{1,2}$-saturated subgraph of P_m^r. Let $e \in E(G_r)$ with $c(e) = k$, $0 \le k \le r$.*
(a) $f(e) \ge 2r - 2 - k$.
(b) $g(e) \le 4r - 2$ if $k = 0$, while $g(e) \le 4r - 2k - 1$ if $k \ge 1$.

Proof. It will be convenient to analyze separately the two cases $c(e) = 0$ and $c(e) \geq 1$.

Suppose first that $c(e) = 0$, and we begin with part b). Then e is an "internal" edge of P_m^r in the sense that both ends of e have degree $2r$ in P_m^r and hence each end is incident on $2r-1$ nonedges of G_r. Thus $g(e) = 4r-2$, giving us part b) when $k = 0$. Next consider part a), still with $c(e) = 0$. Suppose $e = xy \in G_r$ is an edge of dimension d. Since e is internal, for any coordinate s we have that both pairs $\{x(s+), y(s+)\}$ and $\{x(s-), y(s-)\}$ exist in P_m^r. For each $s \neq d$, let $e(+) = x(s+)y(s+) \in E(P_m^r)$. We claim that at least one of $x(s+)$ or $y(s+)$ must be incident on an edge of G_r, call it e_1. For if not, then we have $e(+) \notin E(G_r)$, which together with neither of $x(s+)$, or $y(s+)$ being incident on a G_r edge, implies that $G_r + e(+)$ contains no $K_{1,2}$, contrary to G_r being $K_{1,2}$-saturated. Similarly at least one of $x(s-)$, or $y(s-)$ must be incident on an edge of G_r, call it e_2. So there is a nonedge e_1' of G_r incident on the two edges e and e_1 of G_r, this e_1' being either $xx(s+)$ or $yy(s+)$. Similarly there is a nonedge e_2' of G_r incident on the two edges e and e_2 of G_r, this e_2' being either $xx(s-)$ or $yy(s-)$. So $e_1' \neq e_2'$, and neither e_1' nor e_2' is of dimension d. Therefore e_1' and e_2' together make a contribution of 2 to $f(e)$ corresponding to coordinate s. Summing this contribution over all $r-1$ many dimensions $s \neq d$, we get $f(e) \geq 2r-2$ as required for part a) when $k = 0$. Thus both parts a) and b) of the Lemma are proved when $k = 0$.

Next suppose $c(e) = k$ for some positive integer $1 \leq k \leq r$, again assuming e is d-dimensional. We begin with part a). Again let $e = xy$, so at least one of the ends of e, say x, has k coordinates of value 1 or m. We call these coordinates *extreme* for x, and the remaining $r-k$ coordinates *nonextreme* for x. Let c_1 (resp. c_2) be the number of nonextreme (resp. extreme) coordinates $s \neq d$ for x.

We can express c_1 and c_2 in terms of r and k in the discussion and Claim which follow. For each nonextreme coordinate $s \neq d$, the two pairs of points $x(s+), y(s+)$ and $x(s-), y(s-)$ exist in P_m^r, so the argument given in the preceding paragraph gives a contribution of 2 to $f(e)$ arising from that coordinate. For each extreme coordinate $s \neq d$, the above argument gives a contribution of 1 to $f(e)$ as follows. Here only one of the pairs $x(s+), y(s+)$ and $x(s-), y(s-)$ exists. Then in the preceding paragraph we can guarantee only one of e_1' or e_2' as a nonedge of G_r contributing to $f(e)$, and hence a contribution of 1 to $f(e)$ from that coordinate. Thus we obtain $f(e) \geq 2c_1 + c_2$.

Claim 16 *Let $e = xy \in G_r$ be d-dimensional, and with c_1 and c_2 as just defined. Further let $c(e) = k$, with $b(x) = k$; that is, x has k extreme coordinates.*

a) *If coordinate d is not extreme for x, then $c_1 = r-1-k$ and $c_2 = k$.*

b) *If coordinate d is extreme for x, then $c_1 = r-k$ and $c_2 = k-1$.*

Proof. Consider a), so that d is not extreme for x. Then every coordinate counted in $c(e)$ is also counted in $c_2 \leq c(e)$, so $c_2 = c(e) = k$. The remaining $r-k$ coordinates are nonextreme for x, and among them all but coordinate d are counted in c_1. So $c_1 = r-k-1$, giving part a) of the Claim.

Now consider b), so that d is extreme for x. Then d is the only coordinate counted in $c(e) = k$ which is not counted in $c_2 \leq c(e)$, so $c_2 = c(e) - 1 = k-1$. Again there are $r-k$ nonextreme coordinates s for x, and each satisfies $s \neq d$ since d is extreme. Therefore $c_1 = r-k$, giving part b). ■

We can now apply Claim 16 to obtain the lower bound for $f(e)$ in part a) of the Lemma when $k \geq 1$. Suppose first that coordinate d is not extreme for x. Applying Claim 16a we get $f(e) \geq 2c_1 + c_2 = 2(r-1-k) + k = 2r-2-k$. Now suppose coordinate d is extreme for x. Applying Claim 16b we get $f(e) \geq 2c_1 + c_2 = 2(r-k) + k - 1 = 2r-1-k$. Using the weaker of these lower bounds for $f(e)$, we get the uniform lower bound $f(e) \geq 2r-2-k$. This completes the proof of part a) for all k.

Next we prove b) when $k \geq 1$. For any vertex $v \in G_r$ let $n(v)$ be the number of nonedges e' of G_r in P_m^r incident on v. Thus $g(e) = n(x) + n(y)$. Again take edge e to be d-dimensional, and we continue with the notation c_1 and c_2 defined preceding Claim 16. Recalling the definition of $b(v)$ given before this Lemma, and observing that $|b(x) - b(y)| \leq 1$, we consider the two cases $b(x) = b(y)$ and $b(x) = b(y) + 1$ separately (where interchanging x with y clearly gives the same result). .

Suppose first that $b(x) = b(y) = k$. We determine $n(x)$, the analysis for $n(y)$ being the same. In this case coordinate d cannot be extreme for x, else $b(x) = b(y) + 1$. For each coordinate s counted in c_1 (so

$s \neq d$ and s is not extreme for x) we have a contribution of at most (and exactly) 2 to $n(x)$, coming from the two nonedges $xx(s+)$ and $xx(s-)$, noting that both $x(s+)$ and $x(s-)$ exist in P_m^r since s is not extreme for x. For each coordinate s counted in c_2 (so $s \neq d$ and s is extreme for x) there is a contribution of at most (and exactly) 1 to $n(x)$, coming from the nonedge $xx(s+)$ if $x_s = 1$, or from the nonedge $xx(s-)$ if $x_s = m$. Finally for $s = d$, we recall that d is not an extreme coordinate for x. Therefore there is a contribution of at most (and exactly) 1 to $n(x)$ coming from the nonedge $xx(d-)$ if $y = x(d+)$ or from the nonedge $xx(d+)$ if $y = x(d-)$. Thus $n(x) = 2c_1 + c_2 + 1$. Since d is not an extreme coordinate for x, we have by Claim 16a that $c_1 = r-1-k$ and $c_2 = k$. Thus $n(x) = 2r-k-1$. The same analysis gives $n(y) = 2r-k-1$, where we now use analogous parameters c_1' and c_2', defined as the number of nonextreme (resp. extreme) coordinates $s \neq d$ of y, in the same way we used c_1 and c_2. So finally we obtain

$$g(e) = n(x) + n(y) = 4r - 2k - 2.$$

in the case $b(x) = b(y) = k$.

Suppose now that $k = b(x) = b(y) + 1$. Here coordinate d is extreme for x, but not for y. We compute $n(x)$ using the parameters c_1 and c_2 as we did in the preceding paragraph. There we saw that every coordinate s counted in c_1 contributes at most (and exactly) 2 to $n(x)$, and every coordinate counted in c_2 contributes 1 to $n(x)$. When $s = d$, since d is extreme for x and $b(x) = b(y) + 1$, the only edge of dimension d of P_m^r incident on x is $xy \in E(G_r)$. Thus there is no nonedge of G_r in P_m^r of dimension d incident on x, so the contribution of coordinate d to $n(x)$ is 0. Thus $n(x) = 2c_1 + c_2$. Since d is extreme for x, we have by Claim 16b that $c_1 = r-k$ and $c_2 = k-1$. So we get $n(x) = 2r-k-1$. As for $n(y)$, we use the parameters c_1' and c_2' for y as the analogues of c_1 and c_2 respectively for x (and precisely defined two paragraphs earlier). Since the coordinates of x and y agree in all but coordinate d, we have have $c_1' = c_1$ and $c_2' = c_2$. Also the contributions to $n(y)$ from the coordinates counted in c_1' and c_2' are the same as the corresponding contributions to $n(x)$. For coordinate $s = d$, we get a contribution of 1 to $n(y)$ from the nonedge $yy(d+)$ if $x = y(d-)$ or from nonedge $yy(d-)$ if $x = y(d+)$. So we get $n(y) = 2c_1' + c_2' + 1 = 2r - k$. Altogether then we obtain

$$g(e) = n(x) + n(y) = 4r - 2k - 1.$$

in the case $b(x) = b(y) = k + 1$.

Combining the two cases $b(x) = b(y)$ and $b(x) = b(y) + 1$, we take the larger (i.e. weaker) upper bound for $g(e)$ to cover both cases, and obtain the uniform bound $g(e) \leq 4r-2k-1$ when $c(e) = k \geq 1$. Part b) is thus complete, so the Lemma is proved. ∎

Theorem 17 *$Sat(P_m^r, K_{1,2}) \geq \frac{1}{3}m^r - \frac{1}{3}m^{r-1}$.*

Proof. Let G_r be a $K_{1,2}$-saturated subgraph of P_m^r. We let $E = E(G_r)$ and $\overline{E} = E(P_m^r) - E$ be the set of nonedges of G_r in P_m^r. We continue with the functions $f(e), g(e), c(e)$ defined before Lemma 15, the first two estimated in that Lemma.

We define a weight function on incident edge-nonedge pairs as follows. For any $e \in E$ and $e' \in \overline{E}$, we write $e' \sim e$ to indicate that e and e' are incident edges in P_m^r. Because G_r is $K_{1,2}$-saturated in P_m^r, each nonedge of G_r is incident on either 1 or 2 edges of G_r. So for $e' \sim e$ as above, we let $w(e,e') = 1$ if e is the only edge of G_r incident with e', and $w(e,e') = \frac{1}{2}$ otherwise. It follows that $\sum_{e \in E} \sum_{e' \sim e} w(e,e') = |\overline{E}|$. Let E_0 (resp. E_1) be the set of edges $e \in G_r$ satisfying $c(e) = 0$ (resp. $c(e) > 0$), and let $S_0 = \sum_{e \in E_0} \sum_{e' \sim e} w(e,e')$ and $S_1 = \sum_{e \in E_1} \sum_{e' \sim e} w(e,e')$, so that $|\overline{E}| = S_0 + S_1$. Also let S_e (resp. S_e') be the set of nonedges $e' \sim e$ of G_r in P_m^r satisfying $w(e,e') = \frac{1}{2}$ (resp. $w(e,e') = 1$), so that $g(e) = |S_e| + |S_e'|$.

We claim that $S_0 \leq |E_0|(3r-1)$. Take $e \in E_0$. Since S_e contains all nonedges e' counted in $f(e)$ we get $|S_e| \geq f(e) \geq 2r-2$ by Lemma 15a. Using Lemma 15b we have $|S_e| + |S_e'| = g(e) \leq 4r-2$. It follows that an upper bound for $|S_e'| + \frac{1}{2}|S_e| = \sum_{e' \sim e} w(e,e')$ is obtained by taking $|S_e| = 2r-2$ and $|S_e'| = 2r$. The result is then

$$\textstyle\sum_{e' \sim e} w(e,e') \leq 2r + \frac{1}{2}(2r-2) = 3r-1,$$

and we get $S_0 \le |E_0|(3r-1)$.

We obtain a similar bound for S_1 as follows. For fixed $e \in E_1$, we now have $c(e) = k > 0$. Then applying Lemma 15a we now get $|S_e| \ge f(e) \ge 2r-2-k$. From Lemma 15b we get $|S_e| + |S'_e| = g(e) \le 4r-2k-1$. So an upper bound for $|S'_e| + \frac{1}{2}|S_e|$ is found by taking $|S_e| = 2r-2-k$ and $|S'_e| = 2r-k+1$. So we get

$$\sum_{e' \sim e} w(e,e') \le 2r-k+1+\tfrac{1}{2}(2r-2-k) = 3r - \tfrac{3}{2}k,$$

and we get $S_1 \le |E_1|(3r-\frac{3}{2})$ since $k \ge 1$.

Combining the above bounds we get

$$|\overline{E}| = S_0 + S_1 \le |E_0|(3r-1) + |E_1|(3r-\tfrac{3}{2}) < (|E_0|+|E_1|)(3r-1) = |E|(3r-1).$$

So $|E| \ge \frac{|\overline{E}|}{3r-1}$. Therefore we obtain $|E(P^r_m)| = rm^r - rm^{r-1} = |E| + |\overline{E}| \ge \left(\frac{3r}{3r-1}\right)|\overline{E}|$. Hence $|\overline{E}| \le \left(\frac{3r-1}{3}\right)m^r - \left(\frac{3r-1}{3}\right)m^{r-1}$. So we get

$$|E| \ge rm^r - rm^{r-1} - \left[\left(\tfrac{3r-1}{3}\right)m^r - \left(\tfrac{3r-1}{3}\right)m^{r-1}\right] = \tfrac{1}{3}m^r - \tfrac{1}{3}m^{r-1},$$

completing the proof. ■

4.2 The case $t \ge 3$

We now develop a lower bound for arbitrary $t \ge 3$ and $r \ge 2$ in the following theorem. Actually this result also applies to the case $t = 2$, but the result is weaker for that case than the lower bound of Theorem 17.

Theorem 18 *For $t > 2$ and $r \ge 2$, $Sat(P^r_m, K_{1,t}) \ge m^r\left(\frac{r(t-1)}{4r-t+1}\right) - rm^{r-1}$..*

Proof. Let G be a $K_{1,t}$-saturated subgraph of P^r_m. Let j be such that $t = 2r-j$. Thus the maximum degree in G is $t-1 = 2r-j-1$. We call vertices of degree $2r-j-1$ in G *heavy*, and we call the remaining vertices of G *light*. We let $N = m^r = |V(P^r_m)|$, and M be the number of light vertices in G.

We first obtain upper bounds for $|\overline{E}|$. We will use the degree sum formula applied to the complement of G in P^r_m. Note that every heavy vertex has nonedge degree at most $j+1$, while every light vertex has nonedge degree at most $\Delta(P^r_m) = 2r$. Applying the degree sum formula to the complement of G in P^r_m, we obtain $|\overline{E}| \le \frac{(N-M)(j+1)}{2} + Mr$. Note also that $|\overline{E}| \le (N-M)(j+1)$ since every nonedge is incident on a heavy vertex because G is $K_{1,t}$-saturated in P^r_m.

We now use these upper bounds on $|\overline{E}|$ to lower bound $|E|$. On using $|E| + |\overline{E}| = |E(P^r_m)| = rN - rm^{r-1}$ and the first upper bound for $|\overline{E}|$ in the preceding paragraph, we have

$$|E| \ge (r - \frac{j+1}{2})(N-M) - rm^{r-1}.$$

Using the second upper bound for $|\overline{E}|$ in that paragraph we have $rN - rm^{r-1} = |E| + |\overline{E}| \le (N-M)(j+1) + |E|$, from which we obtain $M \le N\left(1 - \frac{r}{j+1}\right) + \frac{r}{j+1}m^{r-1} + \frac{|E|}{j+1}$. We can substitute this upper bound for M into the preceding lower bound for $|E|$ (since $r - \frac{j+1}{2} > \frac{t-1}{2} > 0$) to obtain

$$|E| \ge (r - \frac{j+1}{2})[\frac{Nr}{j+1} - \frac{|E|}{j+1} - \frac{r}{j+1}m^{r-1}] - rm^{r-1}.$$

Now collecting terms involving $|E|$ and moving them to the left we get

$$|E|(\tfrac{j+1+2r}{2(j+1)}) \ge \tfrac{2r-(j+1)}{2(j+1)}\left(Nr - rm^{r-1}\right) - rm^{r-1}.$$

Solving for $|E|$ and simplifying we get

$$|E| \ge \tfrac{2r-(j+1)}{2r+j+1}\left(Nr - rm^{r-1}\right) - rm^{r-1}\left(\tfrac{2(j+1)}{2r+j+1}\right) = \left(1 - \tfrac{2(j+1)}{2r+j+1}\right)Nr - rm^{r-1}.$$

Now substituting $j = 2r-t$, and noting that $1 - \frac{2(j+1)}{2r+j+1} = \frac{t-1}{4r-t+1}$, we get

$|E| \geq \left(\frac{t-1}{4r-t+1}\right)Nr - rm^{r-1}$, as required. ■

Combining our upper and lower bounds, we get some results which are exact in the leading coefficient.

Corollary 19 *We have the following.*
a) $Sat(P_m^2, K_{1,3}) = \frac{2}{3}m^2 + O(m)$.
b) $Sat(P_m^2, K_{1,4}) = \frac{6}{5}m^2 + O(m)$.
c) $Sat(P_m^r, K_{1,2}) = \frac{1}{3}m^r + O(m^{r-1})$ *for* $r \geq 2$.

Proof. Part a) follows from Theorem 18, letting $t = 3$ and $r = 2$, together with the corresponding upper bound from Theorem 8b. Part b) follows from Theorem 18, letting $t = 4$ and $r = 2$, together with the corresponding upper bound from Theorem 8c. Part c) follows from Theorem 17, together with the corresponding upper bound from Theorem 8a. ■

We show here that the ratio of our upper bound from Theorem 13 to the lower bound from Theorem 18 is less than 2 for $t \geq 3$ and $r = o(m)$, so that our upper bound is within a factor of 2 from optimal under these assumptions. Corollary 19c shows that for $t = 2$ our results are asymptotically exact, so we consider here just $t \geq 3$.

We take t even, as the proof for t odd is nearly identical. Calling this ratio R, we have we have $R = \frac{Am^r + Bm^{r-1}}{Cm^r - Dm^{r-1}}$, where $A = \frac{t}{2} - \frac{4}{5}$, $B = \frac{3t}{10}(1 + o(1))$, $C = \frac{r(t-1)}{4r-t+1}$, $D = r$. We have $R = \frac{A}{C}\left(\frac{1+\frac{B}{Am}}{1-\frac{D}{Cm}}\right)$. Since $3 \leq t \leq 2r$ we have $\frac{D}{C} = \frac{4r-t+1}{t-1} = \frac{4r}{t-1} - 1 \leq 2r - 1$. Since $r = o(m)$, we get $\frac{1}{1-\frac{D}{Cm}} = 1 + o(1)$ as m grows. Also $\frac{B}{A} < 2(1 + o(1))$ as m grows so $1 + \frac{B}{Am} = 1 + o(1)$ as m grows. Thus $R = \frac{A}{C}(1 + o_m(1))$. So it suffices to show that $\frac{A}{C} < 2$. Letting $g(t) = \frac{4r-t+1}{r(t-1)} = \frac{4}{t-1} - \frac{1}{r}$, we have $\frac{A}{C} = \frac{t}{2}g(t) - \frac{4}{5}g(t)$. We have $\frac{t}{2}g(t) = 2(1 + \frac{1}{t-1} - \frac{t}{4r})$. So

$$\frac{A}{C} = 2\left(1 + \frac{1}{t-1} - \frac{t}{4r}\right) - \frac{16}{5(t-1)} + \frac{4}{5r} = 2 - \frac{6}{5(t-1)} - \left(\frac{\frac{t}{2} - \frac{4}{5}}{r}\right) < 2,$$

as required.

References

[1] N. Alon, An extremal problem for sets with applications to graph theory, *J. of Combinatorial Theory - Series A* **40** (1985) 82-89.

[2] A. Arumugam and S. Velammal, Edge domination in graphs, *Taiwanese J. Math.* **2** (1998) 173-179.

[3] B. Bollobas, Determination of extremal graphs by using weights, *Wiss. Z. Techn. Hochsch. Ilmenau* **13** (1967) 419-421.

[4] B. Bollobas, On a conjecture of Erdos, Hajnal and Moon, *The American Mathematical Monthly* **74** (1967) 178-179.

[5] G. Chartrand, L. Lesniak, P. Zhang, *Graphs & Digraphs*, Sixth Edition, Chapman Hall/CRC, 2015.

[6] S. Choi and P. Guan, Minimum critical squarefree subgraph of a hypercube, *Proceedings of the Thrithy-Ninth Southeastern Conf. on Comb., Graph Theory, and Computing*, vol. 189, (2008) 57-64.

[7] B. Currie, J. Faudree, R. Faudree, and J. Schmitt, A survey of minimum saturated graphs, *The Electronic Journal of Combinatorics* Article no. DS19:Oct. 11, 2021, DOI: https://doi.org/10.37236/41.

[8] S. Demyanov and M. Zhulovskii, Tight concentration of star saturation number in random graphs *Discrete Mathematics* **346** (2023) 113572.

[9] P. Erdos, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics* (Cambridge 1983) 1-17, *Academic Press, London*, 1984.

[10] P. Erdos, A Hajnal, and J.W. Moon, A problem in graph theory, *The American Mathematical Monthly* **71** (1964), 1107-1110.

[11] M. Ferrara, M. Jacobson, F. Pfender, and P. Wenger, Graph saturation in multipartite graphs, *Journal of Combinatorics* **7** No. 1 (2016) 1-10.

[12] R. Forcade, Smallest maximal matchings in the graph of the d-dimensional cube, *J. of Comb. Theory* Series B **14(2)** (1973) 153-156.

[13] W. Gan, D. Korandi, B. Sudakov, $K_{s,t}$-saturated bipartite graphs, *European J. of Combinatorics* **45** (2015) 12-20.

[14] R. J. Gould, Developments in saturated graphs, *50 years of Combinatorics, Graph Theory, and Computing*, First edition, Chapman and Hall/CRC (2019) 111-133.

[15] J.R. Johnson and T. Pinto, Saturated subgraphs of the hypercube, *Combinatorics, Probability and Computing* **26** (2017) 52-67.

[16] L. Kaszonyi and Z. Tuza, Saturated graphs with minimal number of edges, *J. of Graph Theory* **10(2)** (1986), 203-210.

[17] W. Klostermeyer and A. Yeo, Edge dominating sets in grids, *J. of Combinatorial Mathematics and Combinatorial Computing* **95** (2015) 99-117.

[18] D. Korandi and B. Sudakov, Saturation in random graphs, *Random Struct. Algorithms* **51** (2017) 169-181.

[19] A. Mohammadian and B. Tayfeh-Rezaie, Star saturation number of random graphs, *Discrete Mathematics* **341** (2018) 1166-1170.

[20] N. Morrison, J. Noel, and A. Scott, Saturation in the hypercube and bootstrap percolation, *Combinatorics, Probability and Computing* **1** (2017) 78-98.

[21] G. Moshkovitz and A. Shapira, Exact bounds for some hypergraph saturation problems, *J. of Comb. Theory* Series B **111** (2015) 242-248.

[22] O. Pikhurko, Extremal Hypergraphs, Ph.D. Thesis, Mathematics, Cambridge University (1999).

[23] W. Wessel, Uber eine Klasse paarer graphen I: Beweis einer vermutung von Erdos, Hajnal, und Moon, *Wiss. Z. Techn. Hochsch. Ilmenau* **12** (1966) 253-256.

[24] W. Wessel, Uber eine Klasse paarer graphen II: Bestimmung de minimalgraphen, *Wiss. Z. Techn. Hochsch. Ilmenau* **13** (1967) 423-426.

[25] Douglas B. West, *Introduction to Graph Theory*, Second Edition, Prentice Hall, 1996, 2001.

[26] Small maximal matchings in random graphs *Theor. Comput. Sci.* **297** (2003) 487-507.

[27] A. Zykov, On some properties of linear complexes (in Russian), *Mat. Sbornik N. S.* **24** (1949), 163-188.