A Solitaire Game Played on 2-Colored Graphs

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Abstract

We introduce a solitaire game played on a graph. Initially one disk is placed at each vertex, one face green and the other red, oriented with either color facing up. Each move of the game consists of selecting a vertex whose disk shows green, flipping over the disks at neighboring vertices, and deleting the selected vertex. The game is won if all vertices are eliminated. We derive a simple parity-based necessary condition for winnability of a given game instance. By studying graph operations that construct new graphs from old, we obtain broad classes of graphs where this condition also suffices, thus characterizing the winnable games on such graphs. Concerning two familiar (but narrow) classes of graphs, we show that for trees a game is winnable if and only if the number of green vertices is odd, and for n-cubes a game is winnable if and only if the number of green vertices is even and not all vertices have the same color. We provide a linear-time algorithm for deciding winnability for games on maximal outerplanar graphs. We reduce the decision problem for winnability of a game on an arbitrary graph G to winnability of games on its blocks, and to winnability on homeomorphic images of G obtained by contracting edges at 2-valent vertices.

1 Introduction and Preliminary Results

We study a solitaire game, which we call *Elimination-Lit Lights Out*, abbreviated as *ELLO*. A game (G, c) of ELLO consists of a graph G together with an initial 2-coloring $c : V(G) \rightarrow \{green, red\}$. Play consists of a sequence of moves, each performed as follows.

(1) A single green vertex x is removed from the current graph.

(2) Vertices in the remaining graph $G - x$ that were neighbors of x change their color.

(3) G is replaced by $G - x$.

The process is repeated, play stopping when no move is possible, i.e., when either no vertices remain (in which case the player has won) or when there are vertices remaining, but all are red (in which case the player has lost, having no game moves available).

A convenient way to play ELLO is to use coins on a drawing of G, showing heads for 'green' and tails for 'red'. For numeric convenience we sometimes use shorthand 0 for 'green' and 1

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for 'red', but the colors green for 'go' and red for 'cannot go' serve as useful reminders of the rules.

Perhaps the first published appearance of these game rules is David Beckwith's problem [8] concerning n identical disks arranged in a circle, each disk having one color on one face, a different color on the other. For ELLO played on the cycle graph C_n as in that problem, it turns out that one can win if and only if initially the number of disks/vertices that are green is even and positive. ELLO played on a path graph P_n appeared in [9] under the name the "coin removal problem". Our interest is in studying ELLO as played on an arbitrary 2-colored graph.

Let p_G and q_G denote the number of vertices and edges (resp.) in a graph G, where we omit the subscript when G is understood. See [20] for other standard graph theoretic notation. A *strategy* for G, or for (G, c), is any linear ordering of $V(G)$. A *winning strategy* for (G, c) is an ordering of $V(G)$ such that the game (G, c) can be won by making moves in this order. The phrase 'π *wins* (G, c)' means that π is a winning strategy for (G, c). Thus π wins (G, c) if and only if each vertex has green as its current color when one begins with (G, c) and attempts to make moves in the order π. A game (G, c) is *winnable* if it has at least one winning strategy; otherwise it is *unwinnable*. Two primary problems that we study concerning this game are:

(1) Efficiently determine whether a given game (G, c) of ELLO is winnable.
(2) Characterize winnable games of ELLO on a given graph.

As an example of the characterization problem, consider the game (G, c) shown in Fig. 1a. It is winnable by moving at vertices B, A, C, D in that order. By contrast, the reader can easily verify that the game (G, c') shown in Fig. 1b is unwinnable. From later results, it will easily follow that a game on this graph G is winnable if and only if the initial number of green vertices is odd. We will find many classes of graphs on which the winnable games are essentially characterized by the parity of the number of green vertices.

Let $d_G(v)$ denote the degree of a vertex v of G. Given a strategy π for G and a vertex v of G, the *predegree of v relative to π*, denoted $d_G(\pi, v)$, is the number of vertices adjacent to v that precede v in π. When G is understood by context, we abbreviate by writing $d(v)$ and $d(\pi, v)$. In the following we use color names 0 and 1 instead of green and red.

Proposition 1.1 *For a strategy π for (G, c), π wins (G, c) if and only if $c(v) + d(\pi, v)$ is even for each vertex v of G.*

Proof. The strategy π is winning if and only if each vertex v has color 0 when played in the ordering π, after having its original color $c(v)$ changed $d(\pi, v)$ many times. The result follows immediately. ∎

Given a strategy π for a graph G, define a vertex coloring c_π of G by

$$c_\pi(v) = \begin{cases} 0 & \text{if } d_G(\pi, v) \text{ is even,} \\ 1 & \text{if } d_G(\pi, v) \text{ is odd.} \end{cases}$$

Corollary 1.2 *(i) For each strategy π for G, (G, c_π) is the unique game on G for which π wins (G, c).*
(ii) A game (G, c) is winnable if and only if there exists a strategy π for which $c_\pi = c$.

We let γ_c denote the number of green vertices (initially) in a game with (initial) coloring c, and let ρ_c denote the number of red vertices, writing simply γ or ρ when the coloring c is clear from context. In playing ELLO on the complete graph K_p, the choice of green vertex at which to move next is immaterial, making optimal play easily analyzed. The next result follows by induction on p, deriving case $p+2$ from case p, upon noticing that making two consecutive moves in a clique changes no vertex colors.

Corollary 1.3 *A game (K_p, c) is winnable if and only if $\gamma_c = \lceil p/2 \rceil$.*

If $\pi = (v_1, v_2, \ldots, v_p)$ is a strategy for G, then the strategy $(v_p, v_{p-1}, \ldots, v_1)$ is denoted by $\overleftarrow{\pi}$. Given a coloring c on $V(G)$, let $\overleftarrow{c}$ denote the coloring obtained from c by changing the color at each odd-degree vertex, leaving the color at each even-degree vertex unchanged. The coloring $\overleftarrow{c}$ is called the *reflection* of coloring c relative to G.

Corollary 1.4 *(i) Given a strategy π for G and a vertex v, $c_\pi(v) = c_{\overleftarrow{\pi}}(v)$ if and only if $d(v)$ is even.*
(ii) Let (G, c) and $(G, \overleftarrow{c})$ be two games on the same graph G. Then (G, c) is winnable if and only if $(G, \overleftarrow{c})$ is winnable. As a special case, if all vertices of G have even degree, then π wins (G, c) if and only if $\overleftarrow{\pi}$ wins $(G, \overleftarrow{c}) = (G, c)$.

Proof. Statement (i) follows from the definitions. For (ii), let (G, c) have winning strategy π. Since $d(\overleftarrow{\pi}, v) = d(v) - d(\pi, v)$, c_π and $c_{\overleftarrow{\pi}}$ differ precisely at vertices of odd degree, so it must be that $\overleftarrow{c} = c_{\overleftarrow{\pi}}$. Therefore, $\overleftarrow{\pi}$ wins $(G, \overleftarrow{c})$, by Corollary 1.2. The converse holds by the same argument, interchanging the roles of c and $\overleftarrow{c}$. ■

We say that a game (G, c) satisfies the *parity condition* if and only if

$$\gamma \equiv p + q \pmod 2.$$

The parity condition is a fundamental necessary condition for a game (G, c) to be winnable, as follows.

Theorem 1.5 *Let G be a graph on p vertices and q edges and let (G, c) be a game on G in which exactly γ of the vertices are green. If (G, c) is winnable, then $\gamma \equiv p + q \pmod 2$.*

Proof. In making a move, a green vertex of some degree d is deleted. This reduces the value of $p+q$ by $d+1$. Since one green vertex is deleted and d others change color, the change in the value of γ is congruent to $d+1 \pmod 2$. Thus, the parity of $p+q-\gamma$ is invariant throughout play. Now, prior to the last move in a winning strategy, we have $p = 1$, $q = 0$, and $\gamma = 1$; thus $p + q - \gamma \equiv 0 \pmod 2$ holds at the end of play and therefore also at the beginning. ■

Proposition 1.6 *Given a strategy π for a game (G, c), the game satisfies the parity condition if and only if there are an even number of vertices v of G for which $c(v) + d(\pi, v)$ is odd.*

Proof. Let $s = \sum_{v \in V(G)}(c(v) + d(\pi, v))$, using 0's and 1's for colors. Since the sum of the predegrees is q, note that $s = \rho + q = p + q - \gamma$. Thus $p + q - \gamma$ is even $\Leftrightarrow$ s is even $\Leftrightarrow$ there are an even number of vertices of G for which $c(v) + d(\pi, v)$ is odd, completing the proof. ■

Two particular colorings of a graph G occur frequently as exceptions to the sufficiency of the parity condition. We introduce those two colorings now, throughout the paper giving these colorings the letter names R and X. Given a graph G, let R denote the all-red coloring of $V(G)$, and let X denote the coloring $\overleftarrow{R}$, noting that (G, R) is of course unwinnable. Note that $X(v) = red = 1$ if $d(v)$ is even and $X(v) = green = 0$ if $d(v)$ is odd. Also, in light of Corollary 1.4, the game (G, X) is unwinnable. The point for now is that games of ELLO with colorings R and X are never winnable, whether or not the games (G, R) and (G, X) satisfy the parity condition. Observe that $(G, R) = (G, X)$ when all vertices of G have even degree. Finally, note that γ_X, being the number of odd-degree vertices, is necessarily even.

Winnable games must satisfy the parity condition. However, in view of the colorings R and X, there are other less fundamental conditions that must also be met for a game to be winnable. For instance, a winnable game on a disconnected graph must break up into winnable games on its components. So that we can discuss claim such as the previous one more precisely, we introduce more notation. For a game (G, c) and an induced subgraph H of G we let $c|_H$ denote the restriction of c to $V(H)$, and call $(H, c|_H)$ the *game induced on H by* (G, c). We have the following.

Corollary 1.7 *Suppose (G, c) is a winnable game of ELLO, with H a component of G. Let c' denote $c|_H$. Then*
(i) $p_H + q_H \equiv \gamma_{c'}$ (mod 2), and
(ii) $\gamma_{c'}$ and $\gamma_{\overleftarrow{c'}}$ are nonzero.
That is, for a game (G, c) to be winnable, necessarily the game induced on each component of G satisfies the parity condition and does not have the all-red coloring R or its reflection X.

Proof. (i) holds, since strategies within a component do not affect play elsewhere. Also, $\gamma \neq 0$ holds, since no first move is available if all vertices are red. By Corollary 1.4(ii) we have that $\gamma_{\overleftarrow{c'}} \neq 0$. ■

Note that Corollary 1.7's necessary conditions for winnability are not sufficient. For example, the all-green game on K_4 satisfies (i) and (ii), yet is unwinnable (by Corollary 1.3). This brings us to a third primary objective for this paper, in addition to the two objectives listed earlier in this section:

(3) Determine large families of connected graphs for which Corollary 1.7's necessary conditions (i) and (ii) for winnability also suffice.

2 ELLO vs. The Game 'Lights Out'

ELLO may remind the reader of a similar game called 'Lights Out'. The latter game is far more well known and well studied, partly because of its successful marketing, and partly because of its direct ties to linear algebra and dominating sets. The 'original' solitaire game called Lights Out was first produced in 1995 by Tiger Electronics as a hand-held electronic game whose board is a 5 by 5 array of squares, where initially each square is either ‘lit’ or ‘unlit’, i.e., ‘on’ or ‘off’. The game’s objective is to turn off all the lights. A move consists of pressing on any square, the effect being to switch the on/off status of that square along with the status of each square with which it shares an edge. By contrast, in ELLO the goal is to eliminate all ‘lights’, where ‘eliminate’ means ‘remove from play’, and a light can be eliminated only when green (‘on’).

Of course, the rules of Lights Out apply perfectly well to 2-colored graphs, where a move at vertex v changes the colors of vertices in v’s closed neighborhood $N[v]$. Letting L denote the set of vertices of G that are initially ‘on’ or ‘lit’, the goal of Lights Out is to find a (minimum) subset S of $V(G)$ for which L is the symmetric difference of the sets $N[v]$ indexed over all $v \in S$. Equivalently, a move at each vertex of S wins at Lights Out if and only if L equals the set of vertices dominated an odd number of times by S. The order in which one makes moves at the vertices of S is immaterial, in contrast with how order matters in ELLO strategy.

For papers concerning Lights Out on m by n grids, see [2], [5], [6], [10] and [19]. For work on Lights Out for trees, cycles and series-parallel graphs, see [2], [3] and [4]. Perhaps the most interesting result for Lights Out on general graphs concerns the case in which initially each vertex is ‘on’; Lights Out with this initial coloring is winnable regardless of the graph involved! For proofs and history of that result, see [11], [12], [14], [15] and [18]. For study of a Lights-Out-related problem called “unbalancing lights”, see [1].

Tiger’s “Lights Out 2000” supports a modulo 3 variation. There are already results (in [13]) on a modulo 3 version of ELLO, those results being the culmination of an undergraduate research project, available at

$$http://www.users.muohio.edu/porterbm/sumj/TOC01.html.$$

The same web site has the undergraduate paper [7] concerning more traditional Lights Out.

In some marketed versions of Lights Out there is a ‘lit’ variation in which one can press a square only when lit. Similarly, in our game, a vertex can be eliminated only when green. Motivated by the restriction that a light can be eliminated only when lit and by the objective of eliminating all lights, we call our game *Elimination-Lit Lights Out*, abbreviated *ELLO*.

3 Constructing Parity-driven Graphs

The general problem of determining whether a given instance of ELLO is winnable seems to be computationally difficult; indeed we believe that the decision problem is NP-complete. Still we manage to find broad classes of graphs for which the necessary conditions of Corollary 1.7 are also sufficient, making the decision problem trivial for such graphs. For a given graph G we introduce the following terms to aid in describing which games (G, c) are winnable. A game or coloring is called *parity-plausible* if it satisfies the parity condition $p + q \equiv \gamma \pmod 2$, *parity-implausible* if it does not, and is called *plausible* if it satisfies the conditions of Corollary 1.7, *implausible* if it does not. A coloring c or game (G, c) is called *trivial* if $c = R$ and *nontrivial* otherwise. A graph G is called *parity-driven* if every nontrivial parity-plausible game on G is winnable, and is *plausibility-driven* if every plausible game on G is winnable.

The parity of $p + q$ is crucial to the parity condition, leading us to say that a graph is an *even-type* graph if $p + q \equiv 0 \pmod 2$, and an *odd-type* graph if $p + q \equiv 1 \pmod 2$.

Note that if c is a parity-plausible coloring on an odd-type graph, then c is not R or X, since γ_R and γ_X are even while γ_c is odd. Thus the concepts of parity-driven and plausibility-driven coincide for connected odd-type graphs.

A parity-driven even-type graph is called an *event*; a parity-driven odd-type graph is called an *oddity*. A graph in which every vertex has even (resp. odd) degree is called an *all-even* graph (resp. *all-odd* graph). The terms 'event' and 'oddity' are named so as to remind us whether such graphs are *even*-type or *odd*-type graphs, and because it may be surprising, eventful, or peculiar to find that a given graph behaves so well that winnability for a game on that graph can be easily determined from knowledge of γ alone.

The following lemma will be used frequently.

Lemma 3.1 *Let G be a graph.*
(i) For $x \in V(G)$, graphs G and $G - x$ are both odd-type or both even-type graphs if and only if $d(x)$ is odd.
(ii) Every event is an all-even graph.
(iii) Any move in a parity-plausible game results in a parity-plausible game.
(iv) If (G, c) is winnable whenever γ_c is odd (resp. γ_c is nonzero and even), then G is an oddity (resp. event).

Proof. Note that $p_{G-x} + q_{G-x} = p_G + q_G - d(x) - 1$, from which (i) easily follows.

For (ii), let G be an event. In particular, G is winnable for every nontrivial parity-plausible c. Now, (G, X) is parity-plausible since γ_X is even. However, (G, X) is unwinnable. It follows that $R = X$. Since by definition $X(v) = red$ if and only if $d(v)$ is even, it follows that G is an all-even graph.

For (iii), see the proof of Theorem 1.5.

For (iv), assume that (G, c) is winnable whenever γ_c is odd. Then G must be an odd-type graph, since otherwise (G, c) would be unwinnable whenever γ_c is odd (by Theorem 1.5). Thus G is an oddity by definition. The second part of (iv) is similar. ■

Note that the only disconnected parity-driven graph is $\overline{K_2}$ (where $\overline{G}$ denotes the graph complement of G), since in a disconnected graph of order greater than 2 it is always possible to find a nontrivial parity-plausible game in which at least one component is all-red. Since $\overline{K_2}$ is an even-type graph, every oddity is connected. Most graphs considered in the remainder of the paper are connected.

As a means of establishing abundant examples of parity-driven graphs, we show how such graphs can be combined to produce larger such graphs. Suppose G has two vertex disjoint induced subgraphs H_1 and H_2 such that $V(G) = V(H_1) \bigcup V(H_2)$, H_1 and H_2 are oddities, and the number of edges in G having one end in H_1 and the other in H_2 is odd. We then say that G *splits into oddities* H_1 and H_2.

Theorem 3.2 *If G splits into oddities H_1 and H_2, then G is an oddity.*

Proof. Let G be a graph which splits as described and let (G, c) be a game on G in which γ_c is odd. By Lemma 3.1(iv), it suffices to show that (G, c) is winnable. Without loss of generality, assume the number of green vertices in H_1 is odd, so the number of green vertices in H_2 is even. Then the induced game on H_1 is winnable. Playing this game on H_1 results in an odd number of individual color changes at vertices in H_2, each corresponding to an edge joining H_1 to H_2. Since H_2 initially contained an even number of green vertices, it now has an odd number after moving successively at each vertex of H_1. Thus, this remaining game on H_2 is winnable, and therefore (G, c) is winnable. ∎

We use Theorem 3.2 to generate various oddities, using an odd number of edges to glue together disjoint oddities H_1 and H_2. However, the case $H_2 = K_1$ deserves special attention, as follows. An ordering $\sigma = v_1, v_2, \ldots, v_p$ of the vertices of a graph G is called an *odd-elimination scheme* for G if the predegrees $d_G(\sigma, v_i)$ are odd for each $i \geq 2$. A graph G is called an *odd-degenerate* graph if it has an odd-elimination scheme, i.e., if K_1 can be formed from G by iteratively deleting a vertex of odd degree in what remains. The term 'odd-degenerate' is as in the standard term 'd-degenerate', wherein the predegrees are required to be d or smaller instead of being odd.

Conjecture 3.3 *Every oddity except for K_1 splits into oddities.*

Conjecture 3.4 *Conjecture 4b: Every oddity is an odd-degenerate graph.*

Proposition 3.5 *(i) Every odd-degenerate graph is an oddity.*
(ii) Every tree is an odd-degenerate graph, and thus also an oddity.
(iii) Conjectures 3.3 and 3.4 are equivalent.

Proof. Clearly K_1 is an oddity. For G an odd-degenerate graph via odd-elimination scheme $v_1, v_2, \ldots, v_p$, it follows by induction that the subgraph induced by the first i vertices splits into the oddity induced by the first $i-1$ vertices and the oddity induced by v_i, so G is an oddity, proving (i). Since every tree can be pruned down to K_1 by iteratively deleting a leaf, (ii) holds.

Clearly Conjecture 3.4 implies 3.3, via a 1 vs. $p-1$ vertex split. For the converse, suppose that every oddity except for K_1 splits into oddities. We prove by induction on the number of vertices that every nontrivial oddity G is an odd-degenerate graph. This holds for K_2. By the induction hypothesis, it suffices to show that G has a vertex v of odd degree such that $G - v$ is an oddity.

Among all ways in which G splits into oddities A and B, consider a way in which $|V(A)|$ is as small as possible. If $|V(A)| > 1$ then A splits into oddities C and D. Without loss of generality, the number of edges of G with one end in C and the other in B is even, and the number of edges of G with one end in D and the other in B is odd. Then $G \setminus V(C)$ is an oddity, since it splits into oddities B and D. But then G splits into oddities C and $G \setminus V(C)$, contradicting that $|V(A)|$ is minimal. Therefore $|V(A)| = 1$, and the vertex of A is the desired vertex v. ∎

Should the above conjectures turn out to be true, it would tell us that the class of oddities has a nice recursive structure. In some sense our search for oddities would be over; we could start with the oddity K_1 and apply Theorem 3.2 repeatedly to build up any oddity, adding on a new vertex and joining it to an odd number of the previous vertices. For those who seek a counterexample to the conjectures, don't bother looking for a nontrivial oddity G in which every vertex has even degree: while such a G could not be an odd-degenerate graph, no such example exists, seen as follows.

Proposition 3.6 *The only all-even oddity is K_1.*

Proof. In an all-even oddity G with 2 or more vertices, consider a coloring c which assigns the color green to only one vertex. Then (G, c) is unwinnable, since the sole green vertex must be first in any winning strategy π, so the last vertex v of π fails to satisfy the condition that $c(v) + d(\pi, v)$ is even, as required by Proposition 1.1 of any winning strategy. ■

For the standard binomial random graph model $G(n, p)$ with edge-probability p with $0 < p < 1$, it is conjectured in [17] that almost all odd-type graphs are odd-degenerate. In support of that conjecture it is proved there that in $G(n, 1/2)$ a random graph is odd-degenerate over 60% of the time, given that the graph is an odd-type graph. Thus it follows for us that most odd-type graphs are oddities, partially accounting for our success in the rest of the paper in showing that a multitude of families of graphs consist of oddities.

Corollary 3.7 *Every complete bipartite graph $K_{m,n}$ is a parity-driven graph, being an odd-degenerate oddity when m or n is odd, and being an event when m and n are even.*

Proof. Suppose m is odd. Then $K_{m,1}$ is odd-degenerate, by Proposition 3.5(ii). Moreover, since $K_{m,n}$ can be constructed from $K_{m,1}$ by repeatedly adding a vertex and joining it by an odd number of edges to the previous graph (by joining it to each of the original m leaves of $K_{m,1}$), we have that $K_{m,2}, K_{m,3}, \ldots, K_{m,n}$ are odd-degenerate. By symmetry, the same is true when n is odd.

Now suppose m and n are even and let $(K_{m,n}, c)$ be a nontrivial parity-plausible game. Make a first move at any green vertex. The resulting game is parity-plausible (Lemma 3.1(iii)) and its graph is an oddity (by the previous case). Since oddities are winnable for all parity-plausible colorings, our original game is winnable, continuing on from our first move. Thus $K_{m,n}$ is an event, since it is a parity-driven graph and is an even-type graph. ■

The *join* of graphs G and H is the graph formed from the disjoint union of G and H by adding an edge from each vertex of G to each vertex of H. In the case where G and H have odd order, the number of added edges is odd, yielding the following corollary.

Corollary 3.8 *If G and H are odd order oddities, then $G \vee H$ is an oddity.*

The following is a generalization of Theorem 3.2.

Theorem 3.9 *Let G be an oddity with $V(G) = \{v_1.v_2, \ldots, v_p\}$ and let $\{H_i\}_{i=1}^p$ be a collection of oddities. Construct a new graph G' from the disjoint union $\bigcup_{i=1}^p H_i$ by adding edges as follows. The number of added edges with one end in $V(H_j)$ and the other in $V(H_k)$ is odd if $v_j v_k \in E(G)$ and is even (possibly 0) if $v_j v_k \notin E(G)$. Then any G' so constructed is an oddity.*

Proof. Let (G', c') be any game on such G', with $\gamma_{c'}$ odd. By Lemma 3.1(iv), it suffices to show that (G', c') is winnable. Define a second game (G, c) by

$$c(v_i) = \begin{cases} \textit{green} & \text{if an odd number of vertices of } H_i \text{ are green under } c', \\ \textit{red} & \text{if an even number of vertices of } H_i \text{ are green under } c'. \end{cases}$$

Since $\gamma_{c'}$ is odd, the number of subgraphs H_i containing an odd number of green vertices is odd. Therefore γ_c must be odd. Hence (G, c) is winnable since G is an oddity. Let π be a winning strategy for (G, c). Renumber the vertices of G, if necessary, so that $\pi = (v_1, v_2, \ldots, v_p)$. We build a winning strategy π' for (G', c') in which the H_i are played out in the order $H_1, H_2, \ldots, H_p$.

Suppose inductively that in playing ELLO starting on (G', c') that we are in mid-play, having so far made moves at the vertices in $H_1, H_2, \ldots, H_{i-1}$ in some order for some i with $1 \le i < p$. Since π is a winning strategy for (G, c), we know that $c(v_i) + d(\pi, v_i)$ is even. By the construction of c, this means that presently there are an odd number of green vertices in H_i, since the parity of the number of green vertices in H_i started out as $c(v_i)$, and in playing out each separate H_j with $j < i$ we have changed the parity of that number if and only if $v_i v_j \in E(G)$. So, since presently there are an odd number of green vertices in H_i, we can continue making ELLO moves by following a winning strategy for the game currently induced on H_i, completing the induction step (where the basis case was when we had made no moves so far). Thus by finite induction we can build the desired winning strategy π' for (G', c'), so G' is an oddity. ■

Let the *Cartesian product* $G \times H$ of graphs G and H be the graph with vertex set $V(G \times H) = V(G) \times V(H)$, where in $G \times H$ vertex (u, v) is adjacent to vertex (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. Since Cartesian products of graphs are constructed in the manner specified by Theorem 3.9, we obtain the following.

Corollary 3.10 *If G and H are oddities and H has odd order, then $G \times H$ is an oddity.*

Proof. We may view $G \times H$ as constructed from G by replacing each vertex of G with a copy of H and replacing each edge xy of G with a matching which joins corresponding points of the two copies of H that replaced x and y. Since this matching has odd size $|V(H)|$, we can apply Theorem 3.9 to conclude that $G \times H$ is an oddity. ■

The Cartesian product process can be iterated, yielding an odd order oddity at each stage so long as the original factor graphs are each of odd order. For example, the Cartesian product of a finite sequence of trees, at most one of which is of even order, is always an oddity. Because the original game of Lights Out and many of its marketed variations are played on grid graphs, we mention the following special cases: the first part is an immediate corollary of Theorem 3.9, while the second part we leave as an exercise. (Hint: append a leaf to a corner of the oddity $P_{m-1} \times P_{n-1}$.)

Proposition 3.11 *(i) If m is odd, then $P_m \times P_n$ is an oddity.*
(ii) If m is odd and n is even, then $C_m \times P_n$ is an oddity.

Given graphs G and H with $V(G) = \{v_1, v_2, \ldots, v_p\}$, the *composition* $G[H]$ is the graph formed by starting with a disjoint union of p copies $H_1, H_2, \ldots, H_p$ of H, and for each edge $v_i v_j$ of G adding in every possible edge having one end in H_i and the other in H_j.

Corollary 3.12 *If G and H are oddities with H of odd order, then $G[H]$ is an oddity.*

While the proofs of Theorems 3.2 and 3.9 involve devising winning strategies by combining strategies for parts of a graph, application of those theorems involves no thought concerning game strategy. We need only keep track of graphs already known to be oddities, and patch them together as dictated in those theorems. Another natural way to avoid discussion of complicated game strategy is to consider favorable games in which, after making one or two arbitrary game moves, it is known in advance that the game remaining is winnable. This motivates the following definitions. A *vertex-deleted subgraph* of a graph G is a subgraph $G - x$ determined by some $x \in V(G)$. A graph is called *special* if each of its vertex-deleted subgraphs is an oddity.

Proposition 3.13 *Every special graph is a parity-driven graph.*

Proof. Let (G, c) be a parity-plausible game on a special graph G. Choose any green vertex x and delete it as a first move in play. The result is a parity-plausible game (by Lemma 3.1(iii)) on an oddity, and is thus winnable. Therefore the original game is also winnable. ■

Since every even-type special graph G is an event and every odd-type special graph H is an oddity, from now on we call such a graph G a *special event* and call such a graph H a *special oddity.* Note from the proof of Proposition 3.13 that for a parity-plausible game on a special graph, one can let an adversary specify any first move, with the assurance of being able to win the game remaining. Special events, like all events, must be all-even graphs (Lemma 3.1(ii)). By contrast, although not every oddity is an all-odd graph, every special oddity is an all-odd graph, which follows from Lemma 3.1(i) on observing that the deletion of any vertex turns the odd-type special oddity into another odd-type graph. Since all-odd graphs have even order, so do special oddities.

Examples: (i) Every cycle C_p is special, since the deletion of any vertex results in a tree (in particular a path), which is an oddity. Since C_p is an even-type graph, it is a special event.
(ii) The Petersen graph P is a special oddity. To see this, verify that P is an odd-type graph and that $P - x$ splits into two induced trees (each an oddity), as shown in Figure 2.
iii) Every even order complete bipartite graph $K_{m,n}$ with $m, n \geq 2$ is a special graph, since upon deletion of any vertex the result is an oddity (by Corollary 3.7).

The next theorem shows how to combine special events in a more flexible way to get larger special events, and similarly for special oddities. It is a special graph analogue of Theorem 3.9 in the form of its statement and proof, and its proof uses Theorem 3.9.

Theorem 3.14 *Let G be a special event [resp. special oddity] with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and let $\{H_i\}_{i=1}^p$ be a collection of vertex disjoint special events [resp. special oddities]. Designate a vertex x_i from each H_i. Construct a new graph G' from the disjoint union $\bigcup_{i=1}^p (H_i - x_i)$ by adding edges as follows. The number of added edges with one end in $V(H_j - x_j)$ and the other in $V(H_k - x_k)$ is odd if $v_j v_k \in E(G)$ and is even (possibly 0) if $v_j v_k \notin E(G)$. If every vertex of G' so constructed has even [resp. odd] degree, then G' is a special event [resp. special oddity].*

Proof. We handle only the statement form in which special events are used to create a larger special event, since the special-oddity-version's proof is essentially identical. Consider any vertex u of such a graph G', where without loss of generality u is in $H_p - x_p$. It suffices to show that $G' - u$ is an oddity, since then not only is G' special, but is an even-type graph by Lemma 3.1(i). Let $H = G' \setminus V(H_p - x_p)$. The graph $G - v_p$ is an oddity, since G is special. Since H is constructed by combining the oddities $H_1 - x_1, H_2 - x_2, \ldots, H_{p-1} - x_{p-1}$ precisely as in Theorem 3.9 by linking them to correspond with the structure of $G - v_p$, graph H is an oddity. Let S denote the set of edges of G' with one end in $H_p - x_p$, and the other end in H.

Consider any vertex z of $H_p - x_p - u$. Let r_1 denote the number of edges of S incident at z, and let r_2 denote the number of edges (0 or 1) from z to x_p in H_p. Then $d_{G'}(z) = d_{H_p}(z) + r_1 - r_2$. Since $d_{G'}(z)$ is even (by hypothesis) and $d_{H_p}(z)$ is even (since H_p is an event), $r_1 \equiv r_2 \pmod 2$. To see that $G' - u$ is an oddity, it suffices to show that it is produced by Theorem 3.9. To apply Theorem 3.9, begin with the oddity $H_p - u$. Now replace x_p by the oddity H, and replace each other vertex of $H_p - u$ by a single vertex (an oddity), i.e., leave the other vertices alone. Add edges between each pair of replacement vertices, other than x_p, exactly as in $H_p - u$. This will satisfy the parity requirements in Theorem 3.9 for how to add new edges. Add edges connecting the vertices of H to each vertex z of $H_p - u$ as prescribed by S, so that the resulting graph is precisely G'. This too satisfies the parity requirements in Theorem 3.9, since we have connected z by an odd number of edges to vertices of H if and only if the edge zx_p was present in $H_p - u$ (since $r_1 \equiv r_2 \pmod 2$). Thus, $G' - u$ is an oddity. ∎

We have used the term 'oddity' for an odd-type parity-driven graph despite the fact that oddities are in abundance (the name 'oddity' serving as a quick reminder of the type). By contrast, events are more rare, if for no other reason than that they must be all-even graphs, so their degree sequences are heavily constrained. Special graphs, whose degree sequences are at least as constrained, are naturally even more rare. We close this section by presenting several general constructions, many of them for producing new families of special events, the rest related to those constructions. For brevity we omit the proofs, but naturally the direct way to show that a graph is special is to show that each of its vertex-deleted subgraphs is an oddity.

Proposition 3.15 *Let G and H be even-type graphs of even order. Suppose that each vertex-deleted subgraph of G and each vertex-deleted subgraph of H is a disjoint union of an odd number of odd order oddities. Then $G \vee H$ is a special event.*

Corollary 3.16 *(i) If G is a special event of even order, then $G \vee \overline{K_{2n}}$ is a special event, and $G \vee \overline{K_{2n-1}}$ is an oddity.*
(ii) Each complete multipartite graph $K_{2,2n_1,2n_2,\ldots,2n_k}$ is a special event.

Proposition 3.17 *If G is a special oddity, then $G \vee K_1$ is a special event.*

Proposition 3.18 *Suppose nonadjacent vertices x, y have the same neighborhood in G, their common degree d being positive and even. Suppose further that $G - x - y$ is a parity-driven graph. Then G is a parity-driven graph, where G is an even-type graph if and only if $G - x - y$ is an even-type graph.*

Corollary 3.19 *(i) If G is connected and with order $p \geq 2$, then $G[\overline{K_2}]$ is an event.*
(ii) Any complete multipartite graph of the form $K_{2n_1,2n_2,\ldots,2n_k}$ is an event.

Based on our success at producing graphs G for which $G - x$ is always an oddity, it is natural to ask whether there exist various graphs G for which $G - x$ is always an event. Or, being more flexible, can a graph G be such that each $G - x$ is a parity-driven graph (i.e., without insisting on the type of $G - x$), without G already being a special graph? K_4 and the complement of P_3 are examples of such graphs, but the following result tells us that no such large graphs are even-type graphs. For brevity, we omit the proof.

Theorem 3.20 *Suppose $|V(G)| \geq 4$ and $G \neq K_4$. Assume further that $G - x$ is an event for each odd degree vertex x, and $G - x$ is an oddity for each even degree vertex x. Then G is a special event (so that in fact G has no odd degree vertices).*

Can an *odd*-type graph G be such that each $G - x$ is a parity-driven graph, without G already being a special graph? The answer is 'yes', as evidenced by the graphs $K_{2m,2n+1}$ for $m, n > 0$. Can an odd-type graph G satisfy the stronger condition that each $G - x$ is an event (where by definition such a G is not special)? Other than $G \cong \overline{K_3}$ the answer is 'no' as follows. By Lemma 3.1(i), every vertex in such a G has even degree. But if $xy \in E(G)$, then y has odd degree in the event $G - x$, contradicting that every vertex in an event has even degree. Therefore G has no edges. It follows easily that $G \cong \overline{K_3}$.

4 Reduction to 2-Connected Graphs

Consider a pair of graphs G and H having exactly one point x in common. Define $G \bigcup_x H$ to be the graph $G \bigcup H$, the x being included in the notation for later purposes as designating the vertex at which G, H have been joined by what is often called *vertex identification*. Given a coloring c of a graph G and a vertex x of G, we let c_x^g denote the coloring of G for which $c_x^g(x) = green$, where $c_x^g(v) = c(v)$ for all vertices v of G other than x. Likewise, c_x^r denotes the coloring of G for which $c_x^r(x) = red$, where $c_x^r(v) = c(v)$ for all vertices v of G other than x. Given a game (G, c) and a vertex x of G, let $plaus(G, c, x)$ denote whichever coloring c_x^g or c_x^r is parity-plausible for G (noting that exactly one of those colorings has γ with the same parity as $p + q$). As notation, if π and π' are two disjoint sequences of vertices in a graph, we let $\pi; \pi'$ denote the sequence formed by concatenating these two sequences, π followed by π'.

Theorem 4.1 *Consider a parity-plausible game $(G_1 \bigcup_x G_2, c)$. For $i = 1, 2$, let $c_i = plaus(G_i, c|_{G_i}, x)$. Then*
(i) $c|_{G_1} = c_1$ if and only if $c_2(x) = green$.
(ii) $(G_1 \bigcup_x G_2, c)$ is winnable if and only if both (G_1, c_1) and (G_2, c_2) are winnable.

Proof. Set $G = G_1 \bigcup_x G_2$. For part (i), let q_1 and q_2 denote the number of edges in G_1 and G_2 respectively, so G has $q_1 + q_2$ edges. Since (G_1, c_1) and (G_2, c_2) and (G, c) are parity-plausible, we have by the parity condition that $\rho_{c_1} \equiv q_1 \pmod 2$, and $\rho_{c_2} \equiv q_2 \pmod 2$, and $\rho_c \equiv q_1 + q_2 \pmod 2$. Let r_1 (resp. r_2) denote the number of vertices of $G_1 - x$ (resp. $G_2 - x$) having color red with respect to c, and define r_x to be 1 if $c(x)$ is red, and 0 if $c(x)$ is green. Then

$r_1 + r_2 + r_x \equiv \rho_c \equiv q_1 + q_2 \equiv \rho_{c_1} + \rho_{c_2} \pmod 2$. So, $r_2 - \rho_{c_2} \equiv \rho_{c_1} - (r_1 + r_x) \pmod 2$. Thus we have that $c|_{G_1} = c_1$ if and only if $\rho_{c_1} \equiv (r_1 + r_x) \pmod 2$ if and only if $r_2 \equiv \rho_{c_2} \pmod 2$ if and only if $c_2(x) = green$, proving (i).

For (ii), suppose (g, C) has winning strategy π, and for $i = 1, 2$ let π_i denote the subsequence of π induced by $V(G_i)$. Apart from x, each vertex of G_i has the same predegree with respect to π_i as it has with respect to π and same color with respect to c_i as with respect to c. Thus π_i is a winning strategy on G_i for one of the colorings $c_x^g|_{G_i}$ and $c_x^r|_{G_i}$, whichever one assigns x the same color as c_{π_i} does. Since c_i is the one parity-plausible coloring among $c_x^g|_{G_i}$ and $c_x^r|_{G_i}$, it must be that π_i is a winning strategy for (G_i, c_i), so both (G_1, c_1) and (G_2, c_2) are winnable.

Conversely, suppose (G_1, c_1) and (G_2, c_2) are winnable, via strategies $\pi_1 = A_1; x; B_1$ and $\pi_2 = A_2; x; B_2$ respectively, for suitable sequences of vertices A_i and B_i, $1 \leq i \leq 2$. It suffices to show that $\pi = A_1; A_2; x; B_1; B_2$ is a winning strategy for (G, c). For each vertex $u \neq x$ of G_i, the predegree of u with respect to π_i is the same as its predegree with respect to π, and $c_i(u) = c(u)$. Hence each vertex $u \neq x$ of G is such that $c(u) + d_G(\pi, u)$ is even. Since c is parity-plausible, by Proposition 1.6 there are an even number of vertices v for which $c(v) + d_G(\pi, v)$ is odd. Thus the remaining vertex x must be such that $c(x) + d_G(\pi, x)$ is even. Thus π wins (G, c), by Proposition 1.1. ■

Using Theorem 4.1, the block-cutpoint tree of a graph, and depth first search on this tree, one can produce an algorithm which, given a parity-plausible game (G, c) on connected G, produces (in $O(e(G))$ time) colorings f_B for each block B of G, with the property that (G, c) is winnable if and only if (B, f_B) is winnable for each block B. We omit the details for brevity. The problem of producing an efficient algorithm for deciding whether an arbitrary game (G, c) is winnable is thus reduced to the same problem under the simplifying assumption that G is 2-connected. As an example, for any game (G, c) in which each block is a clique or a cycle, we can decide in $O(e(G))$ time whether (G, c) is winnable.

For a graph G, suppose that associated to each vertex v of G there is a designated graph G_v such that the family of graphs $\{G_v\}$ is vertex disjoint, where for each v it is the case that $V(G) \bigcap V(G_v) = \{v\}$. Under those circumstances we call the union of graph G with the union of all graphs in $\{G_v\}$ the result of *identifying* the graphs $\{G_v\}$ with G.

Theorem 4.2 *Given any connected graph G, let G' be the result of identifying G with a family $\{G_v\}$ of oddities (where a graph G_v may be K_1). Given any nontrivial parity-plausible game (G', c'), construct a game (G, c) so that*

$$c(v) = \begin{cases} \textit{green} & \textit{if the number of vertices in } G_v \textit{ under } c' \textit{ is odd,} \\ \textit{red} & \textit{if the number of vertices in } G_v \textit{ under } c' \textit{ is even.} \end{cases}$$

Then (G', c') is winnable if and only if (G, c) is winnable.

5 Hypercubes

The well known *hypercube* Q_n of dimension n, for $n \geq 1$, is defined recursively by letting $Q_1 = K_2$ and for $n > 1$ by letting $Q_n = Q_{n-1} \times K_2$. Thus Q_n can be viewed as consisting of two fixed 'copies' of Q_{n-1} joined by a perfect matching in Q_n. A pair of adjacent points x and

y in Q_n, one from each of these two fixed copies of Q_{n-1}, will be called *corresponding*; that is, x *corresponds* to y and vice versa. Our next theorem characterizes winnability on hypercubes, showing the necessary condition, that the coloring be plausible, to be sufficient as well.

Theorem 5.1 *(a) For $n \geq 1$, Q_{2n} is a special event.*
(b) For $n \geq 1$, Q_{2n+1} is plausibility-driven.

Proof. (proof of (a) by induction): $Q_2 \cong C_4$ is clearly a special event. Assume for induction that Q_{2n} is a special event for some $n \geq 1$.

Toward handling Q_{2n+2}, consider any edge uv in Q_{2n+1}. We show that $Q_{2n+1} - u - v$ is an oddity. The cube Q_{2n+1} consists of two $2n$-cube copies Q_{2n}^A and Q_{2n}^B with a matching joining corresponding vertices of the two copies, with $u \in Q_{2n}^A$ and $v \in Q_{2n}^B$. Since Q_{2n}^A and Q_{2n}^B are special events by induction hypothesis, $Q_{2n}^A - u$ and $Q_{2n}^B - v$ are oddities. Since $Q_{2n+1} - u - v$ splits into the oddities $Q_{2n}^A - u$ and $Q_{2n}^B - v$ (joined by an odd number (2^{2n-1}) of edges), it follows by Theorem 3.2 that $Q_{2n+1} - u - v$ is an oddity.

Consider any $x \in V(Q_{2n+2})$. To prove (a), it suffices to show that $Q_{2n+2} - x$ is an oddity. Hypercube Q_{2n+2} can be constructed from two copies Q_{2n+1}^A and Q_{2n+1}^B of Q_{2n+1} with corresponding vertices joined, where $x \in V(Q_{2n+1}^A)$. Let y be any neighbor of x in Q_{2n+1}^A, let y' (resp. x') be the vertex of Q_{2n+1}^B corresponding to y (resp. x), and let z be any neighbor of y' in Q_{2n+1}^B other than x'. See Fig. 3. By the above, $Q_{2n+1}^A - x - y$ and $Q_{2n+1}^B - y' - z$ are both oddities.

Let $H = Q_{2n+2} - x - y - y' - z$. Note that H splits into the oddities $Q_{2n+1}^A - x - y$ and $Q_{2n+1}^B - y' - z$ (joined by an odd number, $2^{2n+1} - 3$, of edges). By Theorem 3.2, H is an oddity. To H, add z and all of its $2n+2$ incident edges except the one to y'. Next, add y' and all of its $2n+2$ incident edges except the one to y. Finally, add y and all of its $2n+2$ incident edges except the one to x. The final result is $Q_{2n+2} - x$, and this is an oddity by successive applications of Theorem 3.2, establishing (a).

(Proof of (b)): Let (Q_{2n+1}, c) be a plausible game, so $c \neq R, X$, where since Q_{2n+1} is all-odd, X is the all-green coloring. It suffices to show that (Q_{2n+1}, c) is winnable. Since c is not monochromatic, there must be adjacent vertices x and y with $c(x) = green$ and $c(y) = red$. Begin play by moving first at x, then y. The result is a parity-plausible game on $Q_{2n+1} - x - y$, that graph shown to be an oddity in the proof of part (a). Thus (Q_{2n+1}, c) is winnable. ∎

6 Winnability and Topological Equivalence

In considering any graph property, a natural problem is to investigate its invariance under homeomorphism (or topological equivalence). For ELLO, the problem can be described as follows. A *thread* in graph G is a path whose internal vertices have degree 2 in G. We say that a graph H is obtained from G by *subdividing* an edge $e = xy$ of G if H is the result of replacing e by a length 2 path x, z, y where z is a new vertex, and leaving all other vertices and edges of G unchanged. Inverting the roles of G and H, we say that H is obtained from G by *suppressing* a vertex z of G if G is obtained from H by subdividing an edge of H using the new vertex z (in the manner of the preceding sentence). We say that graphs G and H are *homeomorphic* if one can be obtained from the other by iteratively applying the operations of subdivision or suppression in any combination. The problem is then to 'reduce' any given game (G, c) to a game (H, c') in which H is homeomorphic to G or to a subgraph of G, and all threads in H are short. Thus we are reduced to a game on a possibly much smaller graph having the same underlying topology as a subgraph of G.

As notation, two games (G, c) and (H, c') are said to have the *same winnability* when (G, c) is winnable if and only if (H, c') is winnable. The next theorem allows us to shorten a thread (obtaining a game with the same winnability) by suppressing any red vertex of degree 2.

Proposition 6.1 *Let (G, c) be a nontrivial parity-plausible game with a red vertex z of degree 2, with $N(z) = \{x, y\}$. Construct H from G by deleting z and changing the adjacency of x and y so that $d_H(x) \equiv d_G(x) \pmod 2$. Let c' denote the restriction of c to $V(H)$. Then (H, c') and (G, c) have the same winnability.*

Proof. Let G, H, x, y, and z be as described. Let π and π' be strategies for G and H respectively, where z is between x and y (though not necessarily consecutive with either x or y) in π, and π' is constructed from π by omitting z. We claim that π wins (G, c) if and only if π' wins (H, c'). For any vertex u of G other than x, y, and z, the predegree condition of Proposition 1.1 holds in π if and only if it holds in π', since z is not adjacent to u. Without loss of generality, suppose x appears before y in π (and consequently also in π'). Then the initial segments of π and π' up to x are identical, so the predegree of x is the same for both. Finally, the predegree of y is the same in π and π' in the case where $H = G - z + xy$, and is reduced by 2 from π to π' in the case where $H = G - z - xy$. Thus, π wins (G, c) if and only if π' wins (H, c'). Now we show that (H, c') and (G, c) have the same winnability. Suppose strategy π wins (G, c). Since z is red, z must appear between x and y in π. Hence by the above, π' wins (H, c'). Conversely suppose π' wins (H, c'). Now create a strategy π for G from π' in the obvious way; placing z anywhere between x and y. By the above, this π wins (G, c). ∎

Corollary 6.2 *Any nontrivial parity-plausible game (G, c) can be reduced to a game (H, c') having the same winnability such that H is homeomorphic to a subgraph of G, and every thread of H has length at most 3 with all its internal vertices being green.*

Proof. Suppose for (G, c) that some thread of G has 3 successive green degree 2 vertices x, z, y. Let H be the graph obtained from G by successively suppressing x and y. Let $c' = c|_{G-x-y}$. It is not hard to verify that (G, c) and (H, c') have the same winnability. Applying this observation along with 6.1 to (G, c) repeatedly to any threads of length exceeding 3, the result follows. ∎

Following up on the results of section 3, we consider the effect of subdivision on parity-driven graphs, where we find in the next theorem that being a parity-driven graph is preserved under homeomorphism.

Proposition 6.3 *Let G be a graph with edge uv and construct H from G be subdividing uv with a vertex z. Then G is a parity-driven graph if and only if H is a parity-driven graph.*

Proof. Observe that G and H are both odd-type graphs or even-type graphs, since H has one more vertex and one more edge than does G. Suppose G is a parity-driven graph and consider a nontrivial parity-plausible game (H, c).

Suppose that $c(z) = red$. Let $c' = c|_{H-z}$. Then c' is also a coloring of $V(G)$, so we can consider the game (G, c'). We see that (G, c') is winnable as follows. Since $\gamma_c = \gamma_{c'}$, certainly γ_c and $\gamma_{c'}$ have the same parity. Further, G and H are both odd-type graphs or even-type graphs, so (G, c') is parity-plausible. Also (G, c') is nontrivial, since (H, c) is nontrivial and $c(z) = red$. Thus (G, c') has some winning strategy $\pi = \{x_1, x_2, \ldots, x_p\}$. Let $u = x_i$ and $v = x_j$ and, without loss of generality, assume $i < j$. Then $\{x_1, x_2, \ldots, x_i, z, x_{i+1}, \ldots, x_p\}$ is a winning strategy for (H, c).

Suppose $c(z) = green$. Without loss of generality we can assume that either $c(u) = red$ or $c(u) = c(v) = green$. Consider the game (G, c'') where $c''(u) \neq c(u)$ and $c''(x) = c(x)$ for each $x \neq u$. Now $c'' \neq R$ since $c''(u)$ and $c''(v)$ cannot both be red. Note that γ_c and $\gamma_{c''}$ have the same parity, since the green vertex z is missing in c'' and vertex u has a different color in c'' versus c. Since $c'' \neq R$, while G and H are both odd-type graphs or even-type graphs, it follows as above that (G, c'') is winnable. Let $\pi = \{x_1, x_2, \ldots, x_p\}$ win for (G, c''), where we let $u = x_i$ and $v = x_j$. The reader can verify that if $i < j$ then $z; \pi$ wins for (H, c), while if $i > j$ then $\pi; z$ wins for (H, c). Therefore, if G is a parity-driven graph then H is a parity-driven graph.

Secondly, suppose H is a parity-driven graph and consider a nontrivial parity-plausible game (G, c^*). Extend c^* to a coloring c for H by letting $c(z) = red$. By Proposition 6.1 we have that (G, c^*) is winnable, so G is a parity-driven graph. ∎

The next result says that being a special event is preserved under subdivision of a thread of length two or more. We omit the fairly straightforward proof.

Proposition 6.4 *Let G be a special event having an edge xy, where the degree of x is 2. Construct H from G by subdividing the edge xy. Then H is a special event.*

7 Maximal Outerplanar Graphs

A maximal outerplanar graph is called a *MOP*. Such a graph is usually drawn as a plane cycle with chords of the cycle dividing its interior into triangular regions, as shown in Fig. 4. The *inner dual* of such a drawing is the ordinary plane dual minus the vertex representing the outer region. The inner dual of a MOP is a tree with maximum degree at most 3. As is often the case, the recursive nature of this inner dual structure allows a reasonable analysis of an arbitrary MOP. While some particularly simple MOP structures have simply stated parity considerations that determine winnability, such is apparently not the case for arbitrary MOPs. In the main result of this section we develop a linear time algorithm to determine whether (G, c) is winnable for a given MOP G. This determination is simpler when the inner dual of G is a path. Such a MOP is called a *linear MOP*, and we discuss these first. We will use the fact that a linear MOP on at least 4 points has exactly two vertices of degree 2.

Theorem 7.1 *Every odd-type linear MOP is an oddity.*

Proof. Let G be an odd-type linear MOP with x and y being its two vertices of degree 2. Since the claim is clear and/or vacuous if $p \leq 3$, we assume $p > 3$, so $xy \notin E(G)$. Begin at x and follow the outer boundary, in either direction, to y. The subgraph induced by the vertices in this portion of the outer boundary is an x, y-path P_1. The remaining vertices also induce a path P_2. We need only observe that G splits into two paths (oddities, joined by an odd number of edges since G is an odd-type grpah) to conclude by Theorem 3.2 that G is an oddity. ■

Proposition 7.2 *Let G be an even-type linear MOP with at least 4 vertices. If (G, c) is a nontrivial parity-plausible game in which either degree 2 vertex is green, then (G, c) is winnable.*

Proof. Let x and y be the vertices of degree 2 and, without loss of generality, let y be green.

Consider the paths P_1 and P_2 as in the proof of the previous result. Since G is an even-type graph and (G, c) is parity-plausible, γ_c is even. Suppose that each of P_1 and P_2 contains an odd number of green vertices. First play out a winning strategy on the oddity P_1. By Lemma 3.1(iii), the remaining game on the oddity P_2 is parity-plausible and hence winnable. Thus (G, c) is winnable overall.

Now suppose each of P_1 and P_2 contains an even number of green vertices. Let $P_1' = P_1 - y$, and let P_2' be the subgraph of G induced by $V(P_2) \bigcup \{y\}$. Since the green vertex y has shifted from one path to the other, each now has an odd number of green vertices. Now apply the above argument with P_1' and P_2' in place of P_1 and P_2. ■

Before we move on to study ELLO on arbitrary MOPs, we mention (proofs omitted) a characterization result for winnability on 'fan graphs', the simplest kind of linear MOPs, and also a characterization result for the closely related 'wheel graphs'.

Proposition 7.3
(a) 'Fan graphs': Let P_n denote the path graph on vertices $y_1, y_2, \ldots, y_n$.

(i) $P_{2n+1} \vee K_1$ is an oddity (by Theorem 7.1).

*(ii) A parity-plausible game $(P_{2n} \vee K_1, c)$ is un*winnable* if and only if $c(y_{2i}) = c(y_{2i+1})$ for $i = 1, 2, \ldots, n-1$ and $c(x) = c(y_1) = c(y_n) = red$ (where x is the vertex from K_1).*

(b) 'Wheel graphs': $C_{2n} \vee K_1$ is an oddity, and $C_{2n+1} \vee K_1$ is an even-type plausibility-driven graph.

Proposition 7.2 leaves incomplete the characterization of winnable games on even-type linear MOPs. In fact we have obtained a complete characterization of winnable games on even-type linear MOPs, but it is sufficiently involved that we choose to omit it here, its statement similar to but far more complicated than that for even-type fan graphs. We speculate that no simple characterization of winnability exists for general MOPs. Still we manage to construct a linear time algorithm (linear in the number of vertices) for deciding winnability for all MOPs.

In preparation for our winnability decision algorithm for MOPs, we present several reduction lemmas allowing us to begin with any game on any MOP and iteratively construct games on smaller order MOPs having the same winnability. The following two lemmas concern the situation shown in Figure 5a, in which adjacent vertices x and y of degree 2 and 3 respectively have vertex a as a common neighbor. Denote the third neighbor of y by b, and let $H = G - x - y$.

Lemma 7.4 (Reduction lemma 1) *Let graph G, subgraph H, and vertices x, y, a and b be as described above and suppose (G, c) is a nontrivial parity-plausible game in which x is red.*
(i) If y is red then the game is winnable if and only if the induced game on H is winnable.
(ii) If y is green then the game is winnable if and only if (H, c') is winnable, where is constructed from the coloring induced on H by changing the color of b.

Proof. Apply Proposition 6.1 to conclude that the given game has the same winnability as the induced game on $G' = G - x - ya$. The red (resp. green) leaf vertex y can always be chosen last (resp. first) in a winning strategy. It follows that the game $(G', c|_{G'})$ has the same winnability as the induced game $(H, c|_H)$ if y is red, and as (H, c') if y is green. The lemma follows. ∎

Lemma 7.5 (Reduction lemma 2) *Let graph G, subgraph H, and vertices x, y, a and b be as described above and suppose (G, c) is a winnable game in which x is green.*
(i) If y is red then there is a winning strategy for (G, c) in which x is first.
(ii) If y is green then there is a winning strategy for (G, c) in which x is last.

Proof. (i) Suppose $c(y) = red$ and let π be a winning strategy for (G, c). Since $c(x) = green$, x cannot appear between its neighbors y and a in π. If it appears *before* both, moving x to the first position (and keeping all other vertices as in π) yields a winning strategy, as this fails to alter the predegree of any vertex. So, in this case, we are done.

If, on the other hand, x *follows* both y and a, apply reasoning similar to the foregoing argument to assume x is last in π. Also since $c(y) = red$, vertex y must appear between its other two neighbors a and b in π. Moreover, we may assume that y and a are consecutive terms of π, as changing the position of y but keeping it between a and b has no effect on predegrees. So, π has one of the following two forms: $A; a; y; B; b; C; x$ or $A; b; B; y; a; C; x$, where A, B, and C represent subsequences of π. In either case, construct a new strategy π' from π by moving x to the beginning and interchanging a and y. The result is either $x; A; y; a; B; b; C$ or $x; A; b; B; a; y; C$ respectively. Since the predegrees of x, y, and a change by an even number (either 0 or 2), and no other predegrees are affected, π' is a winning strategy for (G, c).
(ii) Now suppose $c(y) = green$ and again let π be a winning strategy for (G, c). Consider the game $(G, \overleftarrow{c})$ involving the reflection of coloring c relative to G. Note that x is green and y is red. Apply part (i) of this lemma to conclude that there exists a winning strategy π' for $(G, \overleftarrow{c})$ in which x is first. Reversing this order gives a winning strategy π for (G, c) in which x is last, by Corollary 1.4. ∎

The next two lemmas refer to a situation in which G contains five vertices x, y, z, a, and b for which the neighborhoods of x, y, and z are $N(x) = \{a, y\}$, $N(y) = \{a, b, x, z\}$, and $N(z) = \{b, y\}$, as in Figure 5b. We assume nothing about the adjacency of a and b. Let $H = G - x - y - z$.

Lemma 7.6 (Reduction lemma 3) *Let graph G, subgraph H, and vertices x, y, z, a and b be as described above and suppose (G, c) is a nontrivial parity-plausible game in which $c(z) = red$.*
(i) If $c(x) = c(y) = red$, then (G, c) is unwinnable.

(ii) If $c(x) = c(y) =$ green, then (G, c) has the same winnability as (H, c'), where c' is constructed from the induced coloring on H by changing the color of a.
(iii) If $c(x) \neq c(y)$ then (G, c) has the same winnability as the induced game on H.

When the hypothesis "$c(z) =$ red" of the lemma is replaced by "$c(x) =$ red", the symmetric statements (i.e., interchanging x with z and a with b) hold.

Proof. Let $G'' = G - z - yb$ as shown in Figure 5c and let (G'', c'') be the induced subgame. Then by Proposition 6.1, (G, c) and (G'', c'') have the same winnability.

For (i), observe that in any winning strategy vertex a must precede x and y. But the move at a leaves the component K_2 induced by $\{x, y\}$ with both of its vertices green. This component is then unwinnable.

Parts (ii) and (iii) follow from (G, c) and (G'', c'') having the same winnability, upon applying Theorem 4.1 to the result G'' of performing vertex identification of H and C_3. ■

Lemma 7.7 (Reduction lemma 4) *Let graph G, subgraph H, and vertices x, y, z, a and b be as described above and suppose (G, c) is a winnable game in which $c(x) = c(z) =$ green. Then there is a winning strategy for (G, c) in which x is first.*

Proof. Let π be a winning strategy for (G, c) as described. Since a green vertex of degree 2 must either precede both of its neighbors or follow both neighbors in a winning strategy, x either precedes both y and a or follows both in π. If x precedes y and a, we construct a second winning strategy π' for (G, c) from π by simply moving x to the beginning, yielding the desired claim. Therefore we can assume that x follows both y and a in π. Since moving x to the end of π changes the predegree of no vertex, we can assume that x is last in π. Moreover, the same holds true for z. That is, if z precedes both y and b, there is a winning strategy for (G, c) in which z is first. In this case, simply interchange x and z to construct a winning strategy π' for (G, c) in which x is first. Thus, in all remaining cases, x follows both y and a, and z follows both y and b in π.

Likewise, the green vertex z of degree 2 either precedes both y and b or follows both y and b. In the first case, moving z to the beginning of π changes no vertex predegrees, and in the second case, moving z to just before x near the end of π changes no vertex predegrees. So, we can assume that π has one of the forms $z; A; x$ or $A; z; x$. Likewise, we can assume that y appears consecutively with at least one of a or b in π, since moving y's position one step nearer a or b affects no vertex predegrees. In fact, if y appears between a and b in π, we can assume that it appears consecutive with a.

Case I: Suppose that y does not appear between a and b in π. Let S denote the shortest substring of consecutive terms in π containing a, b and y. Let S' be the string resulting from S by moving y to its end if it was at its front, moving y to its front if it was at its end. The strategy π has form either $z; B; s; C; x$ or $B; S; C; z; x$. If π has the first form, replace π by $\pi' = x; B; S'; C; z$. If π has the second form, replace π by $\pi' = x; z; B; S'; C$. Vertices other than x, y, z are in the same induced order in π vs. π', so the predegrees of vertices other than x, y, z, a and b have not been affected. The reader can verify that the parity of the predegree has not changed for any of x, y, z, a or b in π vs. π', so the condition "$c(v) + d(\pi', v)$ is even" in Proposition 1.1 is assured from that condition for π, making π' a winning strategy in which x is first, as desired.

Case II: Suppose that y appears between a and b in π. Then π has one of the forms at left below. In each case we replace π by the strategy π' indicated at right below. The reader can again verify that each vertex has the same parity for its predegree whether with respect to π or π'. Since x appears first in the winning strategy π' as desired, the claim is proved.

$$\pi = z; B; a; y; C; b; D; x \longrightarrow \pi' = x; z; y; B; a; C; b; D$$

$$\pi = z; B; b; C; y; a; D; x \longrightarrow \pi' = x; z; B; b; C; a; y; D$$

$$\pi = B; a; y; C; b; D; z; x \longrightarrow \pi' = x; B; y; a; C; b; D; z$$

$$\pi = B; b; C; y; a; D; z; x \longrightarrow \pi' = x; B; b; C; a; y; D; z$$

■

We can now use the preceding reduction lemmas to give a linear time algorithm for determining the winnability of any game (G, c) on a maximal outerplanar graph G. Let (G, c) be such a game, σ an initial sequence of game moves in (G, c), and let S denote the set of vertices of σ. Upon playing out the sequence of moves σ in (G, c), we let $(G - S, c \setminus \sigma)$ denote the game remaining (so that $c \setminus \sigma$ is a 2-coloring of $G - S$). Also for a vertex b in G, let $(G, c)_b$ be the game on G obtained from (G, c) by changing the color of b and leaving the colors of all other vertices the same. Finally, when G is a MOP let $idual(G)$ denote the inner dual of G; that is, $idual(G)$ is the graph obtained from the dual of G by deleting the vertex corresponding to the outer region.

The following algorithm operates by replacing a given game on a MOP with a game on a MOP of smaller order, those games having the same winnability. Repeated iterations of this process will produce a game on a MOP small enough that its winnability can be decided by inspection.

Winnability Algorithm on Maximal Outerplanar Graphs
Input: A parity-plausible game (G, c) on a maximal outerplanar graph G
Output: A (correct) decision on whether (G, c) is winnable

1. If the game (G, c) is such that G has at most 4 vertices, then determine by inspection whether (G, c) is winnable, and output "winnable" or "not winnable" as appropriate. Halt.
2. If there is a leaf x' in idual(G) whose neighbor y' has degree 2, then
   2.1) Let the region of G corresponding to x' be bounded by the 3-cycle (x, y, a), in which $d(x) = 2$, $d(y) = 3$. Let the third neighbor of y be b (as in Reduction Lemmas 1 and 2).
   2.2) Replace (G, c) by a game on a smaller graph of same winnability, as follows.
   (i) If $c(x) = red$ and $c(y) = red$, then $(G, c) \leftarrow (G - x - y, c \setminus (x, y))$ (Red. Lemma 1)
   (ii) If $c(x) = red$ and $c(y) = green$, then $(G, c) \leftarrow (G - x - y, c \setminus (x, y))_b$ (Red. Lemma 1)
   (iii) If $c(x) = green$ and $c(y) = red$, then $(G, c) \leftarrow (G - x, c \setminus (x))$ (Red. Lemma 2)
   (iv) If $c(x) = green$ and $c(y) = green$, then $(G, c) \leftarrow (G - x, c|_{G-x})$ (Red. Lemma 2)
   Otherwise,
   2.3) It must be that there exists a pair of leaves in idual(G) having a common neighbor of

degree 3. This is the situation to which Reduction Lemmas 3 and 4 apply. Let vertices x, y, z, a and b be as described prior to Reduction Lemma 3.

2.4) Replace (G, c) by a game on a smaller graph of same winnability, as follows.

(i) If either $c(x) = red$ or $c(z) = red$, say $c(z) = red$, then

(i1) if $c(x) = c(y) = red$, then (G, c) is unwinnable (Red. Lemma 3)

(i2) if $c(x) = c(y) = green$, then $(G, c) \leftarrow (G - x - y - z, (c|_{G-x-y-z})_a$ (Red. Lemma 3)

(i3) if $c(x) \neq c(y)$, then $(G, c) \leftarrow (G - x - y - z, c|_{G-x-y-z})$ (Red. Lemma 3)

(Comment: Symmetrically, if $c(x) = red$, then (G, c) is unwinnable if $c(y) = c(z) = red$, and $(G, c) \leftarrow (G - x - y - z, c|_{G-x-y-z})_b$ if $c(y) = c(z) = green$, and $(G, c) \leftarrow (G - x - y - z, c|_{G-x-y-z})$ if $c(y) \neq c(z)$. Either replacement of (G, c) may be used if both $c(x) = red$ and $c(z) = red$.)

(ii) If $c(x) = c(z) = green$, then $(G, c) \leftarrow (G - x, c \setminus (x))$ (Red. Lemma 4)

3. Go to step 1.

The correctness of this algorithm follows from the application of the reduction lemmas as indicated in the algorithm. In fact one can obtain a $O(n)$ time implementation, where n is the number of vertices in the given outerplanar graph G, by using depth first search on the inner dual tree of G. Again we omit details for the sake of brevity.

Deciding winnability for games on MOPs was somewhat involved. Thus the following easy characterization of winnability on certain MOP related graphs may come as a surprise.

Theorem 7.8 *Suppose G is a MOP. Then upon deleting any edge e from the join of G with a single vertex z, the resulting graph $(G \vee z) - e$ is an oddity.*

Proof. Consider any such edge e, where G is a MOP on n vertices. Being chordal, G has a simplicial elimination ordering $x_n, x_{n-1}, \ldots, x_1$, i.e., an ordering of $V(G)$ such that each x_i has a clique for its neighborhood in the subgraph G_i of G induced by $\{x_1, x_2, \ldots, x_i\}$. Better yet, such an ordering exists for which each x_i among vertices $x_3, x_4, \ldots, x_n$ has degree exactly 2 in G_i, where each G_i is a MOP, and for which both ends of edge e are among x_1, x_2, x_3, z. For each $i \geq 3$ let $H_i = (G_i \vee z) - e$. Subgraph G_3 is a clique K_3, so H_3 is the oddity $K_4 - e$ from Fig. 1. Since each H_{i+1} (for $3 \leq i \leq n - 1$) splits into oddities H_i and x_{i+1} (joined by an odd number - three - of edges), by repeated application of Theorem 3.2 it follows that $H_n = (G \vee z) - e$ is an oddity. ■

8 Concluding Remarks

Two problems arising naturally from our work are the following.

1. Characterize graphs G for which the game (G, c) is winnable if and only if c is plausible. Such odd-type graphs (oddities) are particularly abundant, and hence the characterization problem is of particular interest for oddities. Specifically, is Conjecture 3.4 correct?

2. In Section 3 we reduced the problem of determining whether (G, c) is winnable to the case where G is 2-connected. Still, the complexity of determining the winnability of (G, c) for arbitrary 2-connected graphs remains open. Hence we ask whether the following problem is NP-complete.

Instance: A 2-connected graph G and plausible 2-coloring c of G.
Question: Is (G, c) is winnable?

Beyond these two problems, there are many directions open for further research. For example, one can study the same game when played on directed graphs instead of on graphs, where a move at a vertex x toggles the color at each outneighbor of x, and deletes x. We have some elementary results for this digraph problem. For example, for acyclic digraphs D there is an easy characterization of when (D, c) is winnable. There is reason to believe that proving the winnability decision problem to be NP-complete in the directed graph case may be easier than in the undirected case. We mention that the Parker Brothers game Merlin includes a Magic Square game whose Lights Out-style rules are (effectively) played on a directed graph (see [16] and [19]).

As mentioned earlier, another direction for study has been initiated in [13], in which the colors increase by 1 (mod 3) when they change [or more generally by 1 (mod d)], instead of switching 'mod 2' between just two colors. Even within the mod 3 case, there are choices for how to specify the game rules: at which vertices x can one make a game move, still requiring that x itself gets deleted? In [13] the rule is that a move cannot be made at a vertex of color 0. We hope that the methods and results of this paper lead to further analyses of ELLO and related games.

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