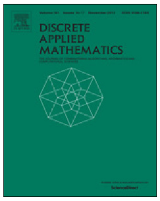
Discrete Applied Mathematics 227 (2017) 84–94

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam



On the bandwidth of the Kneser graph

Tao Jiang [a], Zevi Miller [a,*], Derrek Yager [b]

[a] Department of Mathematics, Miami University, Oxford, OH 45056, USA
[b] Department of Mathematics, University of Illinois, Champaign-Urbana, IL, USA

ARTICLE INFO

Article history:
Received 30 November 2015
Received in revised form 23 November 2016
Accepted 12 April 2017
Available online 29 May 2017

Keywords:
Bandwidth
Kneser graph

ABSTRACT

Let $G = (V, E)$ be a graph on n vertices and $f : V \to [1, n]$ a one to one map of V onto the integers 1 through n. Let *dilation*$(f) = \max\{|f(v) - f(w)| : vw \in E\}$. Define the *bandwidth* $B(G)$ of G to be the minimum possible value of *dilation*(f) over all such one to one maps f. Next define the *Kneser Graph* $K(n, r)$ to be the graph with vertex set $\binom{[n]}{r}$, the collection of r-subsets of an n element set, and edge set $E = \{AB : A, B \in \binom{[n]}{r}, A \cap B = \emptyset\}$. For fixed $r \geq 4$ and n growing we show that

$$B(K(n, r)) = \binom{n-1}{r} + \frac{1}{2}\binom{n-4}{r-1} - \frac{1}{2}\binom{n-1}{r-2} + O(n^{r-4}).$$

1. Introduction

We begin with some notation. Let $[n] = \{1, 2, 3, \ldots, n\}$, which we view as our canonical set of size n. For any finite set S we let $\binom{S}{r}$ be the collection of r-subsets of S. In particular $\binom{[n]}{r}$ will be the collection of r-subsets of $[n]$. For integers $a < b$ we let $[a, b]$ denote the set of integers x satisfying $a \leq x \leq b$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of $[n]$. We say $\mathcal{A}$ is *intersecting* if $A_1 \cap A_2 \neq \emptyset$ for all pairs $A_1, A_2 \in \mathcal{A}$. Further $\mathcal{A}$ is *nontrivial* if $\cap_{A \in \mathcal{A}} A = \emptyset$, and is *trivial* otherwise. The pair of families $\mathcal{A}, \mathcal{B}$ is *cross intersecting* if $A \cap B \neq \emptyset$ for all pairs of sets A, B, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A *matching* of $\mathcal{A}$ is a collection of sets in $\mathcal{A}$ that are pairwise disjoint. For $S \subset [n]$ we let $V(S) = \{x \in [n] : x \in S\}$, and we let $V(\mathcal{A}) = \cup_{S \in \mathcal{A}} V(S)$ (the vertex set of $\mathcal{A}$). We sometimes refer to the sets in $\mathcal{A}$ as *members* of $\mathcal{A}$,. In the sections which follow the introduction, we use small case latin letters $x, y, u, v, \ldots$ to stand for elements of $[n]$, capital letter $A, B, \ldots$ to stand for subsets of size at least 2 from $[n]$ (mostly these will be r-sets), and calligraphy $\mathcal{A}, \mathcal{B}, \ldots$ to stand for collection (or families) of subsets of $[n]$. Apart from these conventions, we use standard graph theoretic or combinatorial notation, as may be found for example in [47]. Additional notation will be defined where it is initially used in the text.

Now define the *Kneser Graph* $K(n, r)$ to be the graph with vertex set $V = \binom{[n]}{r}$, and edge set $E = \{AB : A, B \in \binom{[n]}{r}, A \cap B = \emptyset\}$. We can suppose that $n \geq 2r$ since otherwise $K(n, r)$ has no edges. Clearly $K(n, r)$ has $\binom{n}{r}$ vertices, is regular of degree $\binom{n-r}{r}$, and it can be shown that it is both vertex and edge transitive [39]. The Kneser Graph arises in several examples; $K(n, 1)$ is just the complete graph K_n on n vertices, $K(n, 2)$ is the complement of the line graph of K_n, $K(2n - 1, n - 1)$ is also known as the odd graph O_n, and $K(5, 2)$ is isomorphic to the Petersen graph. The diameter of $K(n, r)$ was shown to be $\lceil \frac{r-1}{n-2r} \rceil + 1$ in [45], and $K(n, r)$ was shown to be Hamiltonian for $n \geq \frac{1}{2}(3r + 1 + \sqrt{5r^2 - 2r + 1})$ in [7].

A longstanding problem on $K(n, r)$ was Kneser's conjecture; that the chromatic number satisfies $\chi(K(n, r)) = n - 2r + 2$ if $n \geq 2r$ and of course $\chi(K(n, r)) = 1$ otherwise. The upper bound is achieved by a simple coloring; color an r-set by its

---

[*] Corresponding author.
E-mail addresses: jiangt@miamioh.edu (T. Jiang), millerz@miamioh.edu (Z. Miller), yager2@illinois.edu (D. Yager).

http://dx.doi.org/10.1016/j.dam.2017.04.008
0166-218X/© 2017 Elsevier B.V. All rights reserved.

largest element if this element is at least $2r$, and otherwise color it by 1. The difficulty was in proving the corresponding lower bound, and this result was first proved by Lovasz [33] using methods of algebraic topology. More elementary, but still topological, proofs were given by Bárány [3] soon after, and by Dol'nikov [13] and Greene [25] later. A mostly combinatorial proof (still with topological elements) was given by Matoušek [36].

Recently some results on a graph labeling problem relating to $K(n, r)$ appeared in the literature [30]. Let $G = (V, E)$ be a graph on n vertices and $f : V \to C_n$ a one to one map of the vertices of G to the cycle C_n on n vertices. Let $|f| = \min\{dist_{C_n}(f(u), f(v)) : uv \in E\}$, where $dist_{C_n}$ denotes the distance function on C_n; that is, $dist_{C_n}(x, y)$ is the mod n distance between x and y when we view the vertices of C_n as the integers mod n. Now let $s(G) = \max\{|f|\}$, where the maximum is taken over all such one to one maps f. It is shown in [30] that $s(K(n, 2)) = 3$ when $n \geq 6$, that $s(K(n, 3)) = 2n - 7$ or $2n - 8$ for n sufficiently large, and that for fixed $r \geq 4$ and n sufficiently large we have $\frac{2n^{r-2}}{(r-2)!} - \frac{(\frac{7}{2}r-2)n^{r-3}}{(r-3)!} - O(n^{r-4}) \leq s(K(n, r)) \leq \frac{2n^{r-2}}{(r-2)!} - \frac{(\frac{7}{2}r-3.2)n^{r-3}}{(r-3)!} + o(n^{r-3})$.

This paper considers the following related well known graph labeling problem. Let $G = (V, E)$ be a graph on n vertices. Now consider $f : V \to [1, n]$ a one to one map, and let $dilation(f) = \max\{|f(v) - f(w)| : vw \in E\}$. Define the *bandwidth* $B(G)$ of G to be the minimum possible value of $dilation(f)$ over all labelings f.

There is an extensive literature on the bandwidth of graphs and related labeling problems (see [8] and [12] for surveys). Apart from its intrinsic interest as a combinatorial problem, bandwidth has connections to other areas in pure and applied mathematics. Its relevance to Ramsey theory and extremal problems was shown in [1,41,6], and [5].

In applied directions, the study of graph bandwidth was motivated by matrix problems in numerical analysis. Here consider a symmetric nxn and $0 - 1$ matrix M. Define $\beta(M)$ to be the minimum integer b such that every nonzero entry of M is located in the set of entries $\{M_{ij} : i - b \leq j \leq i + b, 1 \leq i \leq n\}$ of M. Now given a permutation π of $[n]$, consider the matrix M_π obtained from M by applying π simultaneously to the columns and to the rows of M. The minimum of $\beta(M_\pi)$ over all permutations π is called the bandwidth $B(M)$ of M. Consider the graph G whose adjacency matrix is the original M, making the assumption that the diagonal entries of M are 0. Then it is straightforward to see that $B(G)$ is the same as $B(M)$, where the labelings of G correspond to the permutations π applied to the rows and columns of M. Now the interest in $B(M)$ arises because certain operations on matrices (like Choleski factorization of nonsingular matrices, see [43]) require less space and can be speeded up when the bandwidth of the matrix is small. More recent applications of bandwidth have appeared in the context of information retrieval in browse hypertext (see [44]).

There are general upper and lower bounds as well as exact formulas for $B(G)$ for certain graph classes in terms of natural graph parameters like maximum degree and diameter among others [8] and [12]. We will return to these shortly in the context of the Kneser graph. For example, exact formulas are known for $B(G)$ when G is a path, cycle, complete graph, complete multipartite graph, complete k-ary tree, grid, or hypercube (again, see the above surveys); the formula in the last three cases involving nontrivial arguments (see [26] for hypercubes and [9] for grids). There are also the bounds $B(T) \leq \frac{5n}{\log_\Delta(n)}$ when T is a tree [8], and more generally $B(G) \leq \frac{20n}{\log_\Delta(n)}$ when G is a planar graph [5], both having n vertices and bounded maximum degree Δ.

Concerning the algorithmic complexity of the graph bandwidth problem, it was shown that this problem (suitably stated as a decision problem) is NP-complete [42], even when restricted to the class of trees of maximum degree 3 [23]. Let $B^*(G)$ to be the *topological bandwidth* of G. This is defined as the minimum of $B(H)$ over all graphs H which are refinements of G; that is, those H which can be obtained from G by inserting an arbitrary number of points of degree 2 along any of the edges of G. It was shown in [35] that calculating $B^*(G)$ is NP-complete. But in contrast to the NP-completeness of bandwidth for the class of trees of maximum degree three, it was shown independently in [35] and [38] that topological bandwidth is polynomial time solvable for this class of trees, in fact in time $O(nlog(n))$ in the former paper and in time $O(n)$ in the latter paper, where n is the number of vertices in a tree of maximum degree three. Next define a "caterpillar" to be a tree for which the removal of all leaves results in a path P. Any refinement C^* of C must then be an edge disjoint union of P^*, the refinement of P in C^*, together with a collection of "hairs" of C^*. Each hair h is a refinement in C^* of a length 1 path in C joining a point on P to a leaf of C. In [37] it was shown that $B(C)$ is polynomial time computable when C is a caterpillar, and independently in [2] it was shown that $B(C^*)$ is polynomial time computable when C^* is a caterpillar refinement with all hairs having length 1 or 2. To cap off the previous two results it was shown in [40] that the bandwidth problem for refinements of caterpillars with hair length at most 3 is NP-complete.

Concerning approximation algorithms, it was shown in [11] that the problem of approximating bandwidth on arbitrary graphs to within a constant factor is NP-complete, even when we restrict to the class of refinements of caterpillars. Again for this class, in [16] a polynomial time $O(\frac{log(n)}{log(log(n))})$-approximation algorithm is given, and a $(1 + \epsilon)$-approximation algorithm is given which runs in time $2^{O(\sqrt{\frac{n}{\epsilon}})}$. Now let $0 < \delta < 1$, and define a graph G on n vertices to be δ-dense if the minimum degree of G is at least δn, where n is the number of vertices in G. In [32] a randomized algorithm of running time $n^{O(1/\delta)}$ was given which for any δ-dense graph G produces a labeling f of G satisfying $dilation(f) \leq 3B(G)$ with high probability.

Some computational approaches to the bandwidth problem have been proposed. Two older heuristics can be found in [10] and [24]. A recent paper [46] obtains both lower and upper computational bounds for bandwidth in graphs. The lower bound is based on a new lower bound for the minimum cut problem, which the authors obtain by strengthening a known semidefinite programming relaxation of the quadratic assignment problem. The upper bound is a heuristic based on the Cuthill–McKee algorithm [10] and yields improved upper bounds. Computational results are given for the bandwidth of the Kneser graphs $K(n,2)$, $5 \leq n \leq 8$, and $K(n,3)$, $7 \leq n \leq 10$, achieving the exact value for $B(K(5,2))$ and the currently best known upper bounds for examples from several classes of graphs (including Hamming graphs and Johnson graphs, in addition to Kneser graphs) with up to 216 vertices.

Our main result is the following.

Theorem 1.1. *Let $r \geq 4$ be fixed integer. As n grows we have*

$$B(K(n,r)) = \binom{n-1}{r} + \frac{1}{2}\binom{n-4}{r-1} - \frac{1}{2}\binom{n-1}{r-2} + O(n^{r-4}).$$

A calculation shows that

$$\binom{n-1}{r} + \frac{1}{2}\binom{n-4}{r-1} - \frac{1}{2}\binom{n-1}{r-2} + O(n^{r-4}) = \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\frac{n^{r-2}}{(r-2)!} + (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4}).$$

In the sections which follow it will be convenient to replace the expression on the left of the last equation by the somewhat more involved expression on the right, in order to make an easy comparison with the trivial upper bound (see next paragraph) and for convenience in the arguments. In particular, in the rest of the paper we focus on proving that $B(K(n,r))$ is upper bounded and lower bounded by the expression on the right.

We observe that there is the trivial upper bound $B(K(n,r)) \leq \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1}$, as follows. Let $\alpha(G)$ be the maximum possible size of an independent set of vertices in any graph G on N vertices. Then $B(G) \leq N - \lfloor \frac{1}{2}\alpha(G) \rfloor$, achieved by a one to one map $f : V(G) \to [1, N]$ which sends any half of the vertices of a maximum independent set S to $[1, \lfloor \frac{1}{2}\alpha(G) \rfloor]$, the other half of S to $[N - \lceil \frac{1}{2}\alpha(G) \rceil + 1, N]$, and the remainder of $V(G)$ arbitrarily to the rest of the interval $[1, N]$. Now an independent set in $K(n,r)$ is just an intersecting family in $\binom{[n]}{r}$, and by the Erdos, Ko, Rado theorem [15] the maximum size of such a family is $\binom{n-1}{r-1}$. It follows that $B(K(n,r)) \leq \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1}$.

There are also elementary classically known lower bounds on $B(G)$ for arbitrary G (see the surveys [8] and [12]). Below we apply these to lower bounding $B(K(n,r))$, and we will see that these are strictly smaller than our value for $B(K(n,r))$, and thus yield strictly weaker results than the one presented here. Let $M = \binom{n-1}{r} + \frac{1}{2}\binom{n-4}{r-1} - \frac{1}{2}\binom{n-1}{r-2}$, and let p and q be the number of vertices and edges (respectively) in an arbitrary graph G.

Consider first the known lower bound $B(G) \geq p - \frac{1}{2}\sqrt{(2p-1)^2 - 8q} - \frac{1}{2}$. Using $p(K(n,r)) = \binom{n}{r}$ and $q(K(n,r)) = \frac{1}{2}\binom{n}{r}\binom{n-r}{r}$, and replacing the $(2p-1)^2$ term by $4p^2(1+o(1))$ for p large, we obtain the lower bound $B(K(n,r)) \geq \binom{n}{r}[1 - (1+o(1))\sqrt{1 - \frac{\binom{n-r}{r}}{\binom{n}{r}}}\,]$. Since r is fixed we can estimate $\binom{n}{r} = \frac{n^r}{r!}(1+o(1))$. Now using the upper estimate for $\binom{n-r}{r}$ from Lemma 3.5, we get $\frac{\binom{n-r}{r}}{\binom{n}{r}} \leq (1+o(1))[1 - \frac{C_r}{n} + \frac{D_r}{n^2}]$, where C_r and D_r are positive constants that depend only on r. Thus the known lower bound can be upper bounded as $\binom{n}{r}[1 - (1+o(1))\sqrt{1 - \frac{\binom{n-r}{r}}{\binom{n}{r}}}\,] \leq \binom{n}{r}[1 - (1+o(1))\sqrt{1 - [1 - \frac{C_r}{n} + \frac{D_r}{n^2}]}\,] = \binom{n}{r}[1 - (1+o(1))\sqrt{\frac{C_r}{n} - \frac{D_r}{n^2}}\,]$. So it suffices to verify the inequality $\binom{n}{r}[1 - K\sqrt{\frac{C_r}{n} - \frac{D_r}{n^2}}\,] < M$, where K is a constant with $0 < K < 1$. Dividing each side by $\binom{n-1}{r}$, it suffices to show that $(1 + \frac{r}{n-r})[1 - K\sqrt{\frac{C_r}{n} - \frac{D_r}{n^2}}\,] < 1$. This inequality for large n and fixed r follows on taking logarithms and using the bounds $\ln(1+x) < x$ and $\ln(1-x) < -x$ for $0 < x < 1$.

Next consider the "density" bound $B(G) \geq max_{H \subseteq G} \lceil \frac{(|V(H)|-1)}{diam(H)} \rceil$, where $diam(H)$ is the diameter of any subgraph H of G. Now let H be a subgraph of $K(n,r)$ at which the maximum is attained. If $diam(H) \geq 2$, then the right side of the bound is easily at most $\frac{1}{2}\binom{n}{r} < M$ for large n. If $diam(H) = 1$, then any two r-sets (as vertices of H) are disjoint. Thus $|V(H)| \leq \frac{n}{r}$, so again the density expression is less than M.

Another bound is $B(G) \geq \kappa(G)$, where $\kappa(G)$ is the vertex connectivity of G. Since $\kappa(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree in G, it follows that the most that this lower bound can yield for the Kneser graph is $B(K(n,r)) \geq \binom{n-r}{r}$. But trivially $\binom{n-r}{r} < M$.

A non elementary lower bound for $B(G)$ was developed in [31] and shortly afterwards improved in [27]. It is based on the Laplacian eigenvalues of G, that is, the eigenvalues of $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of degrees of G and $A(G)$ is the adjacency matrix of G. Letting $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the Laplacian eigenvalues of an N-vertex graph G, the result in [27] is that $B(G) \geq \lfloor \frac{\lambda_2}{\lambda_n} N \rfloor$.

In applying this bound to $B(K(n,r))$, we first need the distinct eigenvalues of $K(n,r)$ found by Lovasz in [34]. These are $(-1)^t\binom{n-r-t}{r-t}$, $t = 0, 1, \ldots, r$. From this we obtain $\lambda_2(K(n,r)) = \binom{n-r}{r} - \binom{n-r-2}{r-2}$ and $\lambda_N(K(n,r)) = \binom{n-r}{r} + \binom{n-r-1}{r-1}$, where $N = \binom{n}{r}$. Thus after applying the identity $\binom{s}{t} + \binom{s}{t+1} = \binom{s+1}{t+1}$ we obtain $B(G) \geq \frac{\binom{n-r-1}{r} + \binom{n-r-2}{r-1}}{\binom{n-r}{r} + \binom{n-r-1}{r-1}} \equiv A$.

We now outline the calculations which show that $A < M$. Factoring out $\binom{n-r-2}{r-1}$ from the numerator and $\binom{n-r-1}{r-1}$ from the denominator of A, observing that $\frac{\binom{n-r-2}{r-1}}{\binom{n-r-1}{r-1}} = 1 - \frac{r-1}{n-r-1}$, and simplifying, we are reduced to showing that $(1 - \frac{r-1}{n-r-1})(1 - \frac{1}{n})\binom{n}{r}$

Table 1
Overview of the mapping f.

$\mathcal{S}_1'$	$\mathcal{S}_3$	$V(K(n,r)) - \mathcal{S}_1 - \mathcal{S}_2 - \mathcal{S}_3$	$\mathcal{S}_2$	$\mathcal{S}_1''$

$\le M$. On dividing each side by $\binom{n-1}{r}$ and making the obvious cancellations in ratios of binomial coefficients, we are reduced to $(1+\frac{r}{n-r})(1-\frac{r-1}{n-r-1})(1-\frac{1}{n}) < 1+\frac{1}{2}\frac{(n-r-1)(n-r-2)}{r(n-1)(n-2)(n-3)} - \frac{1}{2}\frac{(r-1)r}{(n-r+1)(n-r)}$. Now straightforward manipulation verifies that this last inequality holds for n growing and r fixed.

Our contribution here is to precisely determine $B(K(n,r))$ for fixed r and n growing, up to an $O(n^{r-4})$ error term. In fact our result shows that the actual value of $B(K(n,r))$ is $2\frac{n^{r-2}}{(r-2)!} - (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4})$ less than the trivial upper bound previously stated. With this in mind, we will occasionally state inequalities involving n and r which are true when n is large enough relative to the fixed r. In these cases we will often not state this requirement on n and r explicitly.

Section 2 will present our upper bound construction together with the resulting dilation bound, and Section 3 will prove a matching lower bound (up to an $O(n^{r-4})$ error term). During the research, the evolving results in each section provided an inspiration for results in the other. Still, these two sections can be read independently of each other.

2. The upper bound

In this section we give a construction of a labeling f such that *dilation*(f) will be our upper bound for $B(K(n,r))$. We begin with some notation. For any sequence $\{i_j\}$ of t integers, $t \le r$, in increasing order, $1 \le i_1 < i_2 < \cdots < i_t \le n$, let $\mathcal{S}_{i_1 i_2 \cdots i_t}$ be the collection of all r-sets in $\binom{[n]}{r}$ whose smallest t elements are $i_1, i_2, \ldots, i_t$, so that $|\mathcal{S}_{i_1 i_2 \cdots i_t}| = \binom{n-i_t}{r-t}$. For example, in $K(8,4)$ the collection of 4-sets $\mathcal{S}_{24}$ is given by $\mathcal{S}_{24} = \{2456, 2457, 2458, 2467, 2468, 2478\}$, and in $K(7,3)$ we have $\mathcal{S}_2 = \{234, 235, 236, 237, 245, 246, 247, 256, 257, 267\}$. Occasionally we insert commas between successive i_j for clarity; e.g. $\mathcal{S}_{1,8,10}$ is the collection of r-sets in $\binom{[n]}{r}$ whose smallest three elements are 1,8, and 10.

With this notation, we are now ready for our construction.

Construction of the map $f : V(K(n,r)) \to \left[1, \binom{n}{r}\right]$.

We begin with some notation. Recall that for a family $\mathcal{A} \subset \binom{[n]}{r}$ we let $f(\mathcal{A}) = \{f(A) : A \in \mathcal{A}\}$. Now let J be a subinterval of $[1, \binom{[n]}{r}]$. When at some step of our construction of f we write $f(\mathcal{A}) = J$, we mean of course that f gives a bijection between $\mathcal{A}$ and J, and that unless otherwise specified later, the order of images $\{f(A), A \in \mathcal{A}\}$ in J is arbitrary. For example, let $\mathcal{A}' \subset \mathcal{A}$ be a subfamily of $\mathcal{A}$, and suppose at a later step we specify that $f(\mathcal{A}') = J' \subset J$ for some subinterval J' of J. Then we now regard the updated f as a "refinement" of the original f, with the further understanding that now also $f(\mathcal{A} - \mathcal{A}') = J - J'$. Still, unless otherwise specified later, we view the order of images $\{f(A'), A' \in \mathcal{A}'\}$ in J' and of images $\{f(B), B \in \mathcal{A} - \mathcal{A}'\}$ in $J - J'$ to be arbitrary at this later step.

With this notation, the construction of f which follows can be roughly summarized in this way. Starting from an initial f motivated by the trivial upper bound for $B(K(n,r))$, we proceed through a series of successive refinements to a final map f. This final f will be such that there is a constructed partition $\cup_i \mathcal{A}_i$ of $\binom{[n]}{r}$ into subfamilies $\mathcal{A}_i$ and a constructed partition $\cup_i J_i$ of $[1, \binom{n}{r}]$ into subintervals J_i such that $f(\mathcal{A}_i) = J_i$. By our notation then, for each i the order of images $\{f(A), A \in J_i\}$ is arbitrary.

Step 1: Map half of $\mathcal{S}_1$, denoted by $\mathcal{S}_1'$, to the left extreme of $[1, \binom{n}{r}]$ and the other half of $\mathcal{S}_1$, denoted by $\mathcal{S}_1''$, to the right extreme of $[1, \binom{n}{r}]$. Thus we have the disjoint union $\mathcal{S}_1 = \mathcal{S}_1' \cup \mathcal{S}_1''$, where we let $|\mathcal{S}_1'| = \lfloor \frac{1}{2}|\mathcal{S}_1| \rfloor = \lfloor \frac{1}{2}\binom{n-1}{r-1} \rfloor$ and $|\mathcal{S}_1''| = \lceil \frac{1}{2}|\mathcal{S}_1| \rceil = \lceil \frac{1}{2}\binom{n-1}{r-1} \rceil$, and where the subsets $\mathcal{S}_1'$ and $\mathcal{S}_1''$ of $\mathcal{S}_1$ will be constructed in the steps which follow. Motivated by the idea behind the trivial upper bound for $B(G)$ given earlier, Step 1 says that $f(\mathcal{S}_1') = [1, |\mathcal{S}_1'|]$, and $f(\mathcal{S}_1'') = [\binom{n}{r} - |\mathcal{S}_1''| + 1, \binom{n}{r}]$. Already our map f, regardless of how we complete its construction, satisfies *dilation*$(f) \le \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1}$.

Step 2: Map $\mathcal{S}_2$ just to the left of $\mathcal{S}_1''$, and $\mathcal{S}_3$ just to the right of $\mathcal{S}_1''$. Specifically we let $f(\mathcal{S}_2) = [\binom{n}{r} - |\mathcal{S}_1''| - |\mathcal{S}_2| + 1, \binom{n}{r} - |\mathcal{S}_1''|]$ and $f(\mathcal{S}_3) = [|\mathcal{S}_1'| + 1, |\mathcal{S}_1'| + |\mathcal{S}_3|]$.

After these two steps, the map f so far is pictured in Table 1. Until we define $\mathcal{S}_1', \mathcal{S}_1''$ and give more details on how r-sets in these collections and in $\mathcal{S}_2$ and $\mathcal{S}_3$ are mapped, we are not yet guaranteed an improvement to the bound *dilation*$(f) \le \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1}$. We proceed to give these details in the next steps, by refining Table 1.

Our motivation is to eliminate "long" edges; that is, $AB \in E(K(n,r))$ for which $|f(A) - f(B)|$ is large. The potentially longest edges exist between pairs $A, B \in \binom{[n]}{r}$ where $f(A)$ is near the left end of $f(\mathcal{S}_1')$ and $f(B)$ is near the right end of $f(\mathcal{S}_2)$, or when $f(A)$ is near the right end of $\mathcal{S}_1''$ and $f(B)$ is near the left end of $f(\mathcal{S}_3)$. To prevent such edges, we define suitably large collections $\mathcal{M}_1 \subset \mathcal{S}_1'$ and $\mathcal{M}_2 \subset \mathcal{S}_2$ such that

(a) $f(\mathcal{M}_1)$ is an initial interval of $f(\mathcal{S}_1')$, and $f(\mathcal{M}_2)$ is a final interval of $f(\mathcal{S}_2)$, and

(b) The pattern of intersections between r-sets in $\mathcal{M}_1$ and r-sets in $\mathcal{M}_2$ prevents long edges. Roughly speaking, we require that for any $A \in \mathcal{M}_1$ the smaller is $f(A)$, the longer is the final subinterval of $J \subset f(\mathcal{M}_2)$ such that $\{A\}$ and $f^{-1}(J)$ are cross intersecting families of r-sets.

Table 2
Refinement of Table 1.

$\mathcal{M}_1$	$\mathcal{S}_1' - \mathcal{M}_1$	$\mathcal{N}_2$	$\mathcal{S}_3 - \mathcal{N}_2$	$V(K(n,r)) - \mathcal{S}_1 - \mathcal{S}_2 - \mathcal{S}_3$	$\mathcal{S}_2 - \mathcal{M}_2$	$\mathcal{M}_2$	$\mathcal{S}_1'' - \mathcal{N}_1$	$\mathcal{N}_1$

Symmetrically we define collections $\mathcal{N}_1 \subset \mathcal{S}_1''$ and $\mathcal{N}_2 \subset \mathcal{S}_3$ satisfying
(c) $f(\mathcal{N}_1)$ is a final subinterval of $f(\mathcal{S}_1'')$, and $f(\mathcal{N}_2)$ is an initial subinterval of $\mathcal{S}_3$, and
(d) The pattern of intersections between r-sets in $\mathcal{N}_1$ and r-sets in $\mathcal{N}_2$ prevents long edges. Here the pattern is that for any $A \in \mathcal{N}_1$, the larger is $f(A)$ the longer is the initial subinterval $I \subset f(\mathcal{S}_3)$ such that $\{A\}$ and $f^{-1}(I)$ are cross intersecting families of r-sets.

The plan just described is illustrated in Table 2, and is a refinement of Table 1. The full execution of this plan is shown in Table 3 (with notation explained after Step 4), which shows the final map f in detail. The steps involved in this execution follow in Steps 3 and 4, which follow.

Step 3: Define the families $\mathcal{M}_i$ and $\mathcal{N}_i$, $i = 1, 2$, all four of them from $\binom{[n]}{r}$, and how they are mapped by f as follows. We let $\mathcal{M}_1 = \mathcal{S}_{12} \cup \mathcal{S}_{156} \cup \mathcal{S}_{157} \cup (\mathcal{S}_{15} - (\mathcal{S}_{156} \cup \mathcal{S}_{157})) \cup \mathcal{S}_{189} \cup \mathcal{S}_{1,8,10}$, presented in order of increasing f-value starting from the left end of $[1, \binom{n}{r}]$. Also let $\mathcal{N}_1 = \mathcal{S}_{13} \cup \mathcal{S}_{145} \cup \mathcal{S}_{146} \cup (\mathcal{S}_{14} - (\mathcal{S}_{145} \cup \mathcal{S}_{146})) \cup \mathcal{S}_{167} \cup \mathcal{S}_{168}$, in order of decreasing f-value starting from the right end of $[1, \binom{n}{r}]$.

Next we let $\mathcal{M}_2 = \mathcal{S}_{258} \cup \mathcal{S}_{259} \cup [\mathcal{S}_{25} - (\mathcal{S}_{258} \cup \mathcal{S}_{259})] \cup \mathcal{S}_{235} \cup \mathcal{S}_{236} \cup [\mathcal{S}_{23} - (\mathcal{S}_{235} \cup \mathcal{S}_{236})]$ in order of decreasing f-value starting from the right end of $f(\mathcal{S}_2)$ (that is, from $\binom{n}{r} - |\mathcal{S}_1''|$). Finally we present $\mathcal{N}_2$ in order of increasing f-value starting from the left end of $f(\mathcal{S}_3)$ (that is, starting from $|\mathcal{S}_1'| + 1$) by letting $\mathcal{N}_2 = \mathcal{S}_{346} \cup \mathcal{S}_{347} \cup [\mathcal{S}_{34} - (\mathcal{S}_{346} \cup \mathcal{S}_{347})] \cup \mathcal{S}_{356} \cup \mathcal{S}_{357} \cup [\mathcal{S}_{35} - (\mathcal{S}_{356} \cup \mathcal{S}_{357})]$.

In step 3 we defined and mapped the families $\mathcal{M}_i$ and $\mathcal{N}_i$, $i = 1, 2$. By our plan (summarized in Table 2), we have $\mathcal{M}_1 \subset \mathcal{S}_1'$ and $\mathcal{N}_1 \subset \mathcal{S}_1''$. Thus it remains to define $\mathcal{S}_1' - \mathcal{M}_1$ and $\mathcal{S}_1'' - \mathcal{N}_1$, and to specify the mapping of the r-sets in these two families, as well as the mapping of all remaining r-sets in $V(K(n, r))$, namely, those in $V(K(n, r)) - (\mathcal{S}_1 \cup \mathcal{M}_2 \cup \mathcal{N}_2)$. We do this in the next step.

Step 4: (a) Construct the families $\mathcal{S}_1' - \mathcal{M}_1$ and $\mathcal{S}_1'' - \mathcal{N}_1$ arbitrarily with r-sets from $\mathcal{S}_1$ so as to arrive at the correct sizes for $\mathcal{S}_1'$ and $\mathcal{S}_1''$. That is, we can let $\mathcal{S}_1' - \mathcal{M}_1$ be any subset of $\mathcal{S}_1$ of size $|\mathcal{S}_1' - \mathcal{M}_1|$, and thus having defined $\mathcal{S}_1'$ we let $\mathcal{S}_1'' - \mathcal{N}_1 = \mathcal{S}_1 - (\mathcal{S}_1' \cup \mathcal{N}_1)$.
(b) Map all remaining r-sets arbitrarily subject only to what was specified in Table 2. That is, let $f(\mathcal{S}_1' - \mathcal{M}_1) = [|\mathcal{M}_1| + 1, |\mathcal{S}_1'|]$, $f(\mathcal{S}_1'' - \mathcal{N}_1) = [\binom{n}{r} - |\mathcal{S}_1''| + 1, \binom{n}{r} - |\mathcal{N}_1|]$, and let $f(V(K(n, r)) - (\mathcal{S}_1 \cup \mathcal{M}_2 \cup \mathcal{N}_2)) = [|\mathcal{S}_1'| + |\mathcal{N}_2| + 1, \binom{n}{r} - |\mathcal{S}_1''| - |\mathcal{M}_2|]$.

This completes the construction of our labeling $f : V(K(n, r)) \to [1, \binom{n}{r}]$. It is illustrated in Table 3, with notation in that table clarified in the comments which follow.

We now comment on some features of f as seen in Table 3. There are 29 cells in this table, counting $\mathcal{R} = V(K(n, r)) - (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$ as a single cell using wraparound. We call these cells *blocks* of f.

Each block labeled $\mathcal{S}_{ij}$, $\mathcal{S}_t$, $\mathcal{S}_{ijk}$, or $\mathcal{R}$ in this table stands (with some abuse of notation) for $f(\mathcal{S}_{ij})$, $f(\mathcal{S}_t)$, $f(\mathcal{S}_{ijk})$, or $f(\mathcal{R})$ (respectively), and is an interval of length $|\mathcal{S}_{ij}|$, $|\mathcal{S}_t|$, $|\mathcal{S}_{ijk}|$, or $|\mathcal{R}|$ (respectively) in $[1, \binom{n}{r}]$. By our convention explained in the two paragraphs preceding Step 1 of the construction of f, the order in which individual r-sets contained in $\mathcal{S}_{ij}$ (or in $\mathcal{S}_t$, $\mathcal{S}_{ijk}$, or $\mathcal{R}$) are mapped to their image block is arbitrary. But the relative order in which these blocks are mapped to $[1, \binom{n}{r}]$ is indicated by the left to right order of their appearance in Table 3, where the second row of the table is understood to follow the first row in left to right order. So for example Table 3 specifies that $f(\mathcal{S}_{12}) = [1, |\mathcal{S}_{12}|] = [1, \binom{n-2}{r-2}]$, that $f(\mathcal{S}_{156}) = [|\mathcal{S}_{12}| + 1, |\mathcal{S}_{12}| + |\mathcal{S}_{156}|] = [\binom{n-2}{r-2} + 1, \binom{n-2}{r-2} + \binom{n-6}{r-3}], \ldots.$, and so on.

We also denote certain blocks using symbols like $\mathcal{S}_{ij}^-$ and $\mathcal{S}_{ij}^+$ in Table 3. To explain by example, we use $\mathcal{S}_{15}^-$ to indicate that some part of $\mathcal{S}_{15}$ (in this case, the part $\mathcal{S}_{156} \cup \mathcal{S}_{157}$) has been mapped immediately to the left of $\mathcal{S}_{15}^-$, and that the block $\mathcal{S}_{15}^-$ is the image of the rest of $\mathcal{S}_{15}$; in this case $\mathcal{S}_{15}^- = f(\mathcal{S}_{15} - (\mathcal{S}_{156} \cup \mathcal{S}_{157}))$. The symbols $\mathcal{S}_1'^-$, $\mathcal{S}_3^-$ have a similar meaning. For example $\mathcal{S}_3^-$ indicates that a part of $\mathcal{S}_3$ as been mapped immediately to the left of $\mathcal{S}_3^-$ (the part $\mathcal{S}_{346} \cup \mathcal{S}_{347} \cup \mathcal{S}_{34}^- \cup \mathcal{S}_{356} \cup \mathcal{S}_{357} \cup \mathcal{S}_{35}^- = \mathcal{S}_{34} \cup \mathcal{S}_{35}$), and that $\mathcal{S}_3^-$ is the image of the rest of $\mathcal{S}_3$; in this case $\mathcal{S}_3^- = f(\mathcal{S}_3 - (\mathcal{S}_{34} \cup \mathcal{S}_{35}))$.

Symmetrically $\mathcal{S}_{14}^+$ indicates that a part of $\mathcal{S}_{14}$ is mapped immediately to the right of $\mathcal{S}_{14}^+$ (this part being $\mathcal{S}_{146} \cup \mathcal{S}_{145}$), and that $\mathcal{S}_{14}^+$ is the image of the rest of $\mathcal{S}_{14}$; in this case $\mathcal{S}_{14}^+ = f(\mathcal{S}_{14} - (\mathcal{S}_{146} \cup \mathcal{S}_{145}))$. Analogous meanings are given to the blocks $\mathcal{S}_1''^+$, $\mathcal{S}_{23}^+$, .., and so on. As an example, $\mathcal{S}_1''^+$ is the first (in increasing order in $[1, \binom{n}{r}]$) of seven consecutive blocks whose union is the image of $\mathcal{S}_1''$.

Our next goal is to prove an upper bound *dilation*(f) for the map f just constructed. A first step toward this goal is to show how we use cross intersecting families to prevent long edges. This will be expressed in the lemma which follows, and which is preceded by some notation.

Let $f : V(K(n, r)) \to [1, \binom{n}{r}]$ be a one to one map, and $\mathcal{A} \in \binom{[n]}{r}$ a collection of r-subsets of $[n]$. For collections $\mathcal{F}, \mathcal{G}$ in $\binom{[n]}{r}$ we say that $\mathcal{G}$ is a *right blocker* of $\mathcal{F}$ if
(a) $f(\mathcal{G}) = [u, \binom{n}{r}]$ for some $\frac{1}{2}\binom{n}{r} < u < \binom{n}{r}$. In particular $f(\mathcal{G})$ is a terminal subinterval of $[1, \binom{n}{r}]$.
(b) $\mathcal{F}$ and $\mathcal{G}$ are cross intersecting; that is, $A \cap B \neq \emptyset$ for any $A \in \mathcal{F}$, $B \in \mathcal{G}$. In particular, $AB \notin E(K(n, r))$ for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

For any collection $\mathcal{F} \subset V(K(n, r))$ and map f as above, let $\Gamma(\mathcal{F}) = \max\{|f(A) - f(B)| : A \in \mathcal{F} \text{ or } B \in \mathcal{F}, AB \in E(K(n, r))\}$. In our construction we analyze $\Gamma(\mathcal{F})$ when $f(\mathcal{F})$ is an interval of integers. The next lemma gives the obvious upper bound on $\Gamma(\mathcal{F})$ in this case.

Table 3
The mapping f in detail, obtained by refining Table 2.

$\mathcal{S}_{12}$	$\mathcal{S}_{156}$	$\mathcal{S}_{157}$	$\mathcal{S}_{15}^-$	$\mathcal{S}_{189}$	$\mathcal{S}_{1,8,10}$	$\mathcal{S}_1'^-$	$\mathcal{S}_{346}$	$\mathcal{S}_{347}$	$\mathcal{S}_{34}^-$	$\mathcal{S}_{356}$	$\mathcal{S}_{357}$	$\mathcal{S}_{35}^-$	$\mathcal{S}_3^-$	$\mathcal{R}$
$\mathcal{R}$	$\mathcal{S}_2^+$	$\mathcal{S}_{23}^+$	$\mathcal{S}_{236}$	$\mathcal{S}_{235}$	$\mathcal{S}_{25}^+$	$\mathcal{S}_{259}$	$\mathcal{S}_{258}$	$\mathcal{S}_1''^+$	$\mathcal{S}_{168}$	$\mathcal{S}_{167}$	$\mathcal{S}_{14}^+$	$\mathcal{S}_{146}$	$\mathcal{S}_{145}$	$\mathcal{S}_{13}$

Lemma 2.1. *Let $f : V(K(n,r)) \to [1, \binom{n}{r}]$ be a one to one map. Suppose $\mathcal{G}$ is a right blocker for $\mathcal{F}$, where $f(\mathcal{F}) = [x, y] \subset [1, \binom{n}{r}]$. Then $\Gamma(\mathcal{F}) \le max\{y - 1, \binom{n}{r} - (|\mathcal{G}| + x)\}$.*

Proof. Consider an edge AB, $A \in \mathcal{F}$. If $f(B) < f(A)$, then since $f(A) \le y$ and $f(B) \ge 1$ we get $|f(A) - f(B)| \le y - 1$ in this case. If $f(B) > f(A)$, then since $B \notin \mathcal{G}$ we have $f(B) \le \binom{n}{r} - |\mathcal{G}|$ while $f(A) \ge x$. It follows that $|f(A) - f(B)| \le \binom{n}{r} - (|\mathcal{G}| + x)$. ■

Now we can prove our upper bound on *dilation*(f).

Theorem 2.2. *Let $r \ge 4$ be a fixed positive integer. Then for large n we have*

$$B(K(n,r)) \le \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\frac{n^{r-2}}{(r-2)!} + (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4}).$$

Proof. We use the terms ("block", $\mathcal{S}_{ij}$, etc.) and explanation of Table 3 as given after Step 4 of the constructions of f above.

Let $L(n,r) = \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\frac{n^{r-2}}{(r-2)!} + (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4})$, the right side of the inequality in the theorem. Observe that *dilation*(f) is the maximum of $\Gamma(\mathcal{F})$ over all blocks $\mathcal{F}$ of f in Table 3. Thus it suffices to show that $\Gamma(\mathcal{F}) \le L(n,r)$ for each such block $\mathcal{F}$.

Start with the special block $\mathcal{F} = \mathcal{R} = V(K(n,r)) - (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$. Note first that $|\mathcal{S}_3| = \binom{n-3}{r-1} < \binom{n-2}{r-1} = |\mathcal{S}_2|$, and $||\mathcal{S}_1'| - |\mathcal{S}_1''|| \le 1$. Thus Table 3 shows that the initial subinterval of $[1, \binom{n}{r}]$ consisting of the 14 blocks immediately preceding $\mathcal{R}$ (i.e. the interval $\mathcal{S}_1' \cup \mathcal{S}_3$) is shorter that the final subinterval of $[1, \binom{n}{r}]$ consisting of the 14 blocks immediately following $\mathcal{R}$ (i.e. the interval $\mathcal{S}_1'' \cup \mathcal{S}_2$). Thus $\Gamma(\mathcal{R}) \le \binom{n}{r} - |\mathcal{S}_1'| - |\mathcal{S}_3| = \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - \binom{n-3}{r-1} < L(n,r)$, as required.

Consider now any block $\mathcal{F} \ne \mathcal{R}$. Each such $\mathcal{F}$ is contained in either $[1, \frac{1}{2}\binom{n}{r}]$ or $[\frac{1}{2}\binom{n}{r}, \binom{n}{r}]$. Since $\frac{1}{2}\binom{n}{r} < L(n,r)$, any edge $AB \in E(K(n,r))$, $f(A) < f(B)$, for which $|f(A) - f(B)| \ge L(n,r)$ must satisfy $f(A) \in [1, \frac{1}{2}\binom{n}{r}]$ and $f(B) \in [\frac{1}{2}\binom{n}{r}, \binom{n}{r}]$. We are thus reduced to showing $\Gamma(\mathcal{F}) \le L(n,r)$ for each block $\mathcal{F}$ with $f(\mathcal{F}) = [x,y] \subset [1, \frac{1}{2}\binom{n}{r}]$, $\mathcal{F} \ne \mathcal{R}$. There are 14 such blocks, in fact the leftmost 14 blocks in the first row of Table 3. Since $y < \frac{1}{2}\binom{n}{r} < L(n,r)$, by Lemma 2.1 we are reduced to showing that for each of these blocks $\mathcal{F}$ there is a right blocker $\mathcal{G}$ of $\mathcal{F}$ such that $|\mathcal{G}| + x \ge \frac{1}{2}\binom{n-1}{r-1} + 2\frac{n^{r-2}}{(r-2)!} - (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4})$. Let $B(n,r)$ be the right side of the last inequality.

For each block $\mathcal{M}$ of f let $\widehat{\mathcal{M}}$ denote the terminal subinterval of $[1, \binom{n}{r}]$ beginning with $\mathcal{M}$, viewed as a union of blocks. As an example, we can use Table 3 to see that $\widehat{\mathcal{S}_{259}} = \{\mathcal{S}_{259} \cup \mathcal{S}_{258} \cup \mathcal{S}_1''^+ \cup \mathcal{S}_{168} \cup \mathcal{S}_{167} \cup \mathcal{S}_{14}^+ \cup \mathcal{S}_{146} \cup \mathcal{S}_{145} \cup \mathcal{S}_{13}\}$. With this notation, for each block $\mathcal{F} \subset [1, \frac{1}{2}\binom{n}{r}]$, Table 4 gives a right blocker $\mathcal{G}$ of $\mathcal{F}$ (listed directly below $\mathcal{F}$ in the table) in the form $\mathcal{G} = \widehat{\mathcal{M}}$ for some block $\mathcal{M}$ from Table 3. For example, the right blocker of $\mathcal{S}_{157}$ given by Table 4 is $\widehat{\mathcal{S}_{235}}$.

To verify that these $\mathcal{G}$ are indeed right blockers, we need to check that $\mathcal{F}$ and the corresponding $\mathcal{G} = \widehat{\mathcal{M}}$ are cross intersecting families. As notation for doing this, for any block $\mathcal{F}$ of f, let $\mathcal{I}(\mathcal{F}) = \bigcap_{A \in \mathcal{F}} A$, the intersection of all r-sets in $\mathcal{F}$. In particular $\mathcal{I}(\mathcal{S}_{ij}) = \{i, j\}$, $\mathcal{I}(\mathcal{S}_{ijk}) = \{i, j, k\}, \ldots,$ and so on. It suffices to show that for each block $\mathcal{B}$ appearing in $\widehat{\mathcal{M}}$ we have $\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{F}) \ne \emptyset$. For example, when $\mathcal{F} = \mathcal{S}_{189}$, the corresponding right blocker for $\mathcal{F}$ given in Table 4 is $\mathcal{G} = \widehat{\mathcal{S}_{259}}$, and the blocks $\mathcal{B}$ comprising $\mathcal{G}$ are given in the preceding paragraph. Examining each of these blocks $\mathcal{B}$ (in left to right order) for the condition $\mathcal{I}(\mathcal{B}) \cap \mathcal{I}(\mathcal{F}) \ne \emptyset$ we get $9 \in \mathcal{I}(\mathcal{S}_{259}) \cap \mathcal{I}(\mathcal{S}_{189})$, $8 \in \mathcal{I}(\mathcal{S}_{258}) \cap \mathcal{I}(\mathcal{S}_{189}), \ldots, 1 \in \mathcal{I}(\mathcal{S}_{13}) \cap \mathcal{I}(\mathcal{S}_{189})$, as required. We leave to the reader the similar verification that for blocks $\mathcal{F} \ne \mathcal{R}$, $\mathcal{F} \subset [1, \frac{1}{2}\binom{n}{r}]$, we have that $\mathcal{F}$ and the corresponding $\mathcal{G} = \widehat{\mathcal{M}}$ given by Table 4 are cross intersecting families.

Let then $\mathcal{F}$ be any block of f with $f(\mathcal{F}) = [x,y] \subset [1, \frac{1}{2}\binom{n}{r}]$. We now verify the property $|\mathcal{G}| + x \ge B(n,r)$, where $\mathcal{G}$ is the right blocker of $\mathcal{F}$ given in Table 4.

First suppose that $\mathcal{F} = \mathcal{S}_{12}$, so $x = 1$. Table 4 gives the right blocker $\mathcal{G} = \widehat{\mathcal{S}_2^+} = \mathcal{S}_2 \cup \mathcal{S}_1''$. Hence $|\mathcal{G}| + 1 > |\mathcal{G}| = |\mathcal{S}_2| + |\mathcal{S}_1''| = \binom{n-2}{r-1} + \lceil \frac{1}{2}\binom{n-1}{r-1} \rceil > B(n,r)$. Next let $\mathcal{F} = \mathcal{S}_3^-$, where the corresponding right blocker is $\mathcal{G} = \widehat{\mathcal{S}_{13}} = \mathcal{S}_{13}$. Table 3 shows that the left endpoint x of $f(\mathcal{S}_3)$ is $x = |\mathcal{S}_1'| + |\mathcal{S}_{34}| + |\mathcal{S}_{35}| + 1$, so $|\mathcal{G}| + x > \lfloor \frac{1}{2}\binom{n-1}{r-1} \rfloor + \binom{n-4}{r-2} + \binom{n-5}{r-2} + \binom{n-3}{r-2} > B(n,r)$ easily for fixed r and n large.

So we are reduced to considering blocks satisfying $\mathcal{F} \notin \{\mathcal{S}_{12}, \mathcal{S}_3^-\}$. The crucial feature of f which ensures $|\mathcal{G}| + x \ge B(n,r)$ for such blocks $\mathcal{F}$ is that the right blocker $\mathcal{G}$ of $\mathcal{F}$ given in Table 4 satisfies

$$|G| + x = 1 + |\mathcal{S}_1''(or\mathcal{S}_1') \cup \mathcal{S}_{ab} \cup \mathcal{S}_{cd} \cup \mathcal{S}_{rst} \cup \mathcal{S}_{r's't'}|, b + d = 7 \tag{1}$$

for suitable integers $a, b, c, d, r, s, t, r', s', t'$, and where the five sets on the right side have pairwise empty intersection. Suppose for a moment that this property holds for $\mathcal{F}$ and its right blocker $\mathcal{G}$. The using Lemma 3.5 we obtain $|\mathcal{G}| + x = 1 + \frac{1}{2}\binom{n-1}{r-1} + \binom{n-b}{r-2} + \binom{n-d}{r-2} + \binom{n-t}{r-3} + \binom{n-t'}{r-3} = 1 + \frac{1}{2}\binom{n-1}{r-1} + 2\frac{n^{r-2}}{(r-2)!} - (\frac{b+d+r-3-2}{(r-3)!})n^{r-3} + O(n^{r-4}) = B(n,r)$, as required, since $b + d = 7$.

Table 4
Block $\mathcal{F}$ of f, right blocker $\mathcal{G}$ for $\mathcal{F}$.

$\mathcal{F}$	$\mathcal{S}_{12}$	$\mathcal{S}_{156}$	$\mathcal{S}_{157}$	$\mathcal{S}_{15}^{-}$	$\mathcal{S}_{189}$	$\mathcal{S}_{1,8,10}$	$\mathcal{S}_{1}^{\prime -}$	$\mathcal{S}_{346}$	$\mathcal{S}_{347}$	$\mathcal{S}_{34}^{-}$	$\mathcal{S}_{356}$	$\mathcal{S}_{357}$	$\mathcal{S}_{35}^{-}$	$\mathcal{S}_{3}^{-}$
$\mathcal{G}$	$\widehat{\mathcal{S}_{2}^{+}}$	$\widehat{\mathcal{S}_{236}}$	$\widehat{\mathcal{S}_{235}}$	$\widehat{\mathcal{S}_{25}^{+}}$	$\widehat{\mathcal{S}_{259}}$	$\widehat{\mathcal{S}_{258}}$	$\widehat{\mathcal{S}_{1}^{\prime\prime +}}$	$\widehat{\mathcal{S}_{168}}$	$\widehat{\mathcal{S}_{167}}$	$\widehat{\mathcal{S}_{14}^{+}}$	$\widehat{\mathcal{S}_{146}}$	$\widehat{\mathcal{S}_{145}}$	$\widehat{\mathcal{S}_{145}}$	$\widehat{\mathcal{S}_{13}}$

It remains to verify that (1) holds for these $\mathcal{F}$ (and their corresponding $\mathcal{G}$). We do this for three representative cases, and leave the similar and straightforward verification of the others to the reader. Consider first $\mathcal{F} = \mathcal{S}_{157}$. From Table 3 we have $x = 1 + |\mathcal{S}_{12} \cup \mathcal{S}_{156}|$. Table 4 gives the right blocker for $\mathcal{F}$ given by $\mathcal{G} = \widehat{\mathcal{S}_{235}} = \mathcal{S}_{235} \cup \mathcal{S}_{25} \cup \mathcal{S}_1''$. Thus $|\mathcal{G}| + x = 1 + |\mathcal{S}_1'' \cup \mathcal{S}_{25} \cup \mathcal{S}_{12} \cup \mathcal{S}_{156} \cup \mathcal{S}_{235}|$, as required by (1). As a second example let $\mathcal{F} = \mathcal{S}_{1,8,10}$. Table 3 gives $x = 1 + |\mathcal{S}_{12}| + |\mathcal{S}_{15}| + |\mathcal{S}_{189}|$, and Table 4 gives the right blocker $\mathcal{G} = \widehat{\mathcal{S}_{258}} = \mathcal{S}_{258} \cup \mathcal{S}_1''$ of $\mathcal{F} = \mathcal{S}_{1,8,10}$. Thus we obtain $|\mathcal{G}| + x = 1 + |\mathcal{S}_1'' \cup \mathcal{S}_{12} \cup \mathcal{S}_{15} \cup \mathcal{S}_{189} \cup \mathcal{S}_{258}|$, thereby verifying (1) for $\mathcal{F} = \mathcal{S}_{1,8,10}$. Finally consider $\mathcal{F} = \mathcal{S}_{347}$. Using Tables 3 and 4 we have $x = 1 + |\mathcal{S}_1' \cup \mathcal{S}_{346}|$, while the right blocker for $\mathcal{F}$ is $\mathcal{G} = \widehat{\mathcal{S}_{167}} = \mathcal{S}_{167} \cup \mathcal{S}_{14} \cup \mathcal{S}_{13}$. So finally $|\mathcal{G}| + x = 1 + |\mathcal{S}_1' \cup \mathcal{S}_{14} \cup \mathcal{S}_{13} \cup \mathcal{S}_{167} \cup \mathcal{S}_{346}|$, as required by (1). We leave to the reader the similar verification that (1) holds for the remaining blocks in the first row of Table 3 satisfying $\mathcal{F} \notin \{\mathcal{S}_{12}, \mathcal{S}_3\}$, $\mathcal{F} \neq \mathcal{R}$. ■

3. The lower bound

Our goal in this section is to prove the lower bound $B(K(n,r)) \geq \binom{n}{r} - A$, where $A = \frac{1}{2}\binom{n-1}{r-1} + 2\frac{n^{r-2}}{(r-2)!} - (r+2)\frac{n^{r-3}}{(r-3)!} + O(n^{r-4})$; that is, to show that *dilation*$(f) \geq \binom{n}{r} - A$ for any one to one map $f : V(K(n,r)) \to [1, \binom{n}{r}]$.

As notation, for two families of r-sets $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{r}$, let us write $\mathcal{A} \sim \mathcal{B}$ (resp. $\mathcal{A} \nsim \mathcal{B}$) to mean that $\mathcal{A}$ and $\mathcal{B}$ are (resp. are not) cross intersecting. Thus $\mathcal{A} \nsim \mathcal{B}$ means that there exist r-sets $S, T \in \binom{[n]}{r}$ with $S \in \mathcal{A}$ and $T \in \mathcal{B}$ such that $S \cap T = \emptyset$. So $\mathcal{A} \nsim \mathcal{B}$ says that there is an edge $ST \in E(K(n,r))$ with $S \in \mathcal{A}$ and $T \in \mathcal{B}$. Roughly speaking we will be showing that for any one to one map $f : V(K(n,r)) \to [1, \binom{n}{r}]$ there is an initial (resp. final) subinterval I (resp. F) of $[1, \binom{n}{r}]$, with $|I| + |F|$ reasonably small, such that $f^{-1}(I) \nsim f^{-1}(J)$. This forces a "long" edge ST; that is one satisfying $|f(S) - f(T)| \geq \binom{n}{r} - (|I| + |F|)$, and leads to our lower bound on $B(K(n,r))$.

We discuss briefly the relation between our lower bound proof and existing results in the literature on cross intersecting families. Now *dilation*$(f) \geq \binom{n}{r} - p$ is equivalent to $f^{-1}(j) \nsim f^{-1}([\binom{n}{r} - p + j, \binom{n}{r}])$ for some j, $1 \leq j \leq p$. This is in turn equivalent to $f^{-1}([1,j]) \nsim f^{-1}([\binom{n}{r} - p + j, \binom{n}{r}])$ for some j, $1 \leq j \leq A$. In these and also other cases we will be interested in proving $\mathcal{A} \nsim \mathcal{B}$ for certain pairs $\mathcal{A}, \mathcal{B}$ of families of subsets of $[n]$. Many results in the literature ([28,16], and [21] are examples) produce upper bounds on $|\mathcal{A}| + |\mathcal{B}|$ or $|\mathcal{A}||\mathcal{B}|$ whenever $\mathcal{A}$ and $\mathcal{B}$ are cross intersecting. If we could show that the families $\mathcal{A}, \mathcal{B}$ for which we try to prove $\mathcal{A} \nsim \mathcal{B}$ violate these bounds, then we could indeed conclude that $\mathcal{A} \nsim \mathcal{B}$. In the examples just cited, the bounds were too generous for our purposes, and to the best of our knowledge the same holds for the other published results of this type. Indeed in our setting, we will be dealing with cross-intersecting families $\mathcal{A}, \mathcal{B}$ where one of them contains a large matching, which substantially restricts $|\mathcal{A}| + |\mathcal{B}|$.

We mention here a few results from the literature that will be useful. The following extension of the Hilton–Milner theorem on intersecting families to cross-intersecting families was established by Füredi [22].

Theorem 3.1 (*[22]*). *Let n, a, b be positive integers where $n \geq a + b$. Let $\mathcal{A} \subseteq \binom{[n]}{a}$ and $\mathcal{B} \subseteq \binom{[n]}{b}$, and suppose that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting. If $|\mathcal{A}| \geq \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$ and $|\mathcal{B}| > \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$, then there exists an element $x \in [n]$ that lies in all members of $\mathcal{A}$ and $\mathcal{B}$. That is; $\mathcal{A} \cup \mathcal{B}$ is a trivial family.*

Let t be a positive integer. A family $\mathcal{F}$ of sets is said to be *t-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. Erdős, Ko, and Rado [15] proved that for fixed positive integers r, t, where $r \geq t + 1$, there exists $n_0(r,t)$ such that if $n \geq n_0(r,t)$ then the maximum size of an t-intersecting family of r-subsets of $[n]$ is $\binom{n-t}{r-t}$. For $t \geq 15$, Frankl [17] obtained the smallest possible $n_0(r,t)$ for which the statement holds. Wilson [48] obtained the smallest possible $n_0(r,t)$ for all t.

Theorem 3.2 (*[17] for $t \geq 15$, [48] for all t*). *For all $n \geq (r-t+1)(t+1)$, if $\mathcal{F} \subseteq \binom{[n]}{r}$ satisfies that $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$ (i.e. $\mathcal{F}$ is t-intersecting) then $|\mathcal{F}| \leq \binom{n-t}{r-t}$.*

Erdős [14] showed that there exists $n_1(r,p)$ such that for all $n \geq n_1(r,p)$ the maximum size of a family of r-subsets of $[n]$ not containing a matching of size $p+1$ is $\binom{n}{r} - \binom{n-p}{r}$. There has subsequently been a lot of work on determining the smallest $n_1(r,p)$ for which the statement holds (see [4,19,20,29] for instance). The best result among these is due to Frankl [18].

Theorem 3.3 (*[18]*). *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ such that $\mathcal{F}$ contains no matching of size $p+1$, where $n \geq (2p+1)r - p$. Then $|\mathcal{F}| \leq \binom{n}{r} - \binom{n-p}{r}$.*

Note that $\binom{n}{r} - \binom{n-p}{r} < p\binom{n-1}{r-1}$. For our purpose, we will just use the following weakening of Theorem 3.3 that applies to all n.

Lemma 3.4 (*[18]*). *Suppose $\mathcal{F} \subseteq \binom{[n]}{r}$ satisfies $|\mathcal{F}| > p\binom{n-1}{r-1}$. Then $\mathcal{F}$ contains a matching of size $p+1$.*

In fact, Frankl showed that if $\mathcal{F} \subseteq \binom{n}{r}$ contains no $(s+1)$-matching then $|\mathcal{F}| \le s|\delta(\mathcal{F})|$, where $\delta(\mathcal{F})$ denotes the number of distinct $(r-1)$-sets that are contained in edges of $\mathcal{F}$.

The following lemma is straightforward to verify.

Lemma 3.5. *Let n, r, c be integers, where $n \ge r \ge 2$ and $c \le r$. We have*

$$\frac{n^r}{r!} - \frac{c + \frac{r-1}{2}}{(r-1)!} n^{r-1} \le \binom{n-c}{r} \le \frac{n^r}{r!} - \frac{c + \frac{r-1}{2}}{(r-1)!} n^{r-1} + 4r^4 n^{r-2}.$$

We can now prove our lower bound result, starting with some intuition guiding our proof. Recall the idea behind the trivial upper bound. We map half of a large intersecting family to an initial subinterval, and the other half to a terminal subinterval of $[1, \binom{n}{r}]$. Guided by this, we show that (unless our lower bound holds) there must be a reasonably large intersecting family, of which half, call it $\mathcal{A}$, is mapped to an initial interval and the other half $\mathcal{B}$ to a final subinterval. Recall now the collections $\mathcal{M}_i$ and $\mathcal{N}_i$ from the upper bound construction. We show that for any labeling f (not already satisfying our dilation lower bound), either the analogues of the pair of collections $\{\mathcal{M}_1, \mathcal{M}_2\}$ cannot have a too large combined size, or the analogues of the pair $\{\mathcal{N}_1, \mathcal{N}_2\}$ cannot have a too large combined size. That is, we can find a pair of not very long intervals in $[1, \binom{n}{r}]$, one of them an initial subinterval contained in $\mathcal{A}$ and the other an interval just preceding $\mathcal{B}$ which are not cross intersecting, or the same holds with "initial" replaced by "terminal" and the roles of $\mathcal{A}$ and $\mathcal{B}$ interchanged. This is accomplished by extremal set theoretic arguments, and finally averaging.

Theorem 3.6. *Let $r \ge 4$ be a fixed positive integer. Let f be a bijection from $V(K(n, r))$ to $\{1, \dots, \binom{n}{r}\}$. Then for n sufficiently large relative to r we have*

$$\mathit{dilation}(f) \ge \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\frac{n^{r-2}}{(r-2)!} + \frac{r+2}{(r-3)!} n^{r-3} - 9r^4 n^{r-4}.$$

Proof. Let

$$N = \left\lceil \frac{1}{2}\binom{n-1}{r-1} \right\rceil - r\binom{n-2}{r-2}$$

and

$$\mathcal{A} = \left\{ D \in \binom{[n]}{r} : 1 \le f(D) \le N \right\} \text{ and } \mathcal{B} = \left\{ D \in \binom{[n]}{r} : \binom{n}{r} - N + 1 \le f(D) \le \binom{n}{r} \right\}.$$

We may assume n to be sufficiently large relative to r so that $\mathcal{A}$ and $\mathcal{B}$ are disjoint.

Claim 1. *There exists an element $x \in [n]$ that lies in all members of $\mathcal{A} \cup \mathcal{B}$; that is $\mathcal{A} \cup \mathcal{B}$ is trivial.*

Proof of Claim 1. Let $\mathcal{A}' = \{D \in \binom{[n]}{r} : f(D) \le r\binom{n-2}{r-2}\}$ and $\mathcal{B}' = \{D \in \binom{[n]}{r} : f(D) \ge \binom{n}{r} - r\binom{n-2}{r-2} + 1\}$. Then $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$. If $\mathcal{A}'$ and $\mathcal{B}$ are not cross-intersecting then there exist A, B in $\binom{[n]}{r}$ with $f(A) \le r\binom{n-2}{r-2}$ and $f(B) \ge \binom{n}{r} - N + 1$ such that $AB \in E(K(n, r))$, which yields $\mathit{dilation}(f) \ge |f(x) - f(y)| \ge \binom{n}{r} - N + 1 - r\binom{n-2}{r-2} \ge \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} + 1$, and the theorem is proved. Hence we may suppose that $\mathcal{A}'$, $\mathcal{B}$ are cross-intersecting. Similarly, $\mathcal{A}$, $\mathcal{B}'$ are cross-intersecting. For sufficiently large n, we have $|\mathcal{A}'| = r\binom{n-2}{r-2} > \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$ and $|\mathcal{B}| \ge \frac{1}{2}\binom{n-1}{r-1} - r\binom{n-2}{r-2} - 1 > \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$. By Theorem 3.1, there exists an element $x \in [n]$ that lies in all members of $\mathcal{A}'$ and $\mathcal{B}$. By a similar argument, there exists an element $y \in [n]$ that lies in all members of $\mathcal{A}$ and $\mathcal{B}'$. Suppose $x \ne y$. Then x, y both lie in all members of $\mathcal{A}'$, which is impossible since there are only $\binom{n-2}{r-2}$ many r-subsets of $[n]$ containing both x and y, while $|\mathcal{A}'| = r\binom{n-2}{r-2} \ge 4\binom{n-2}{r-2}$. Hence $x = y$. So the element x lies in all r-subsets of $\mathcal{A} \cup \mathcal{B}$. ■

Without loss of generality, we may assume that element 1 lies in all members of $\mathcal{A} \cup \mathcal{B}$. Let $\mathcal{S}_1 = \{D \in \binom{[n]}{r}, 1 \in D\}$ and $\overline{\mathcal{S}_1} = \{D \in \binom{[n]}{r} : 1 \notin D\}$. Then $\mathcal{A} \cup \mathcal{B} \subseteq \mathcal{S}_1$, $|\mathcal{S}_1| = \binom{n-1}{r-1}$, and $|\overline{\mathcal{S}_1}| = \binom{n-1}{r}$.

Claim 2. *Let $\mathcal{A}_0 \subseteq \mathcal{S}_1$ and $\mathcal{X} \subseteq \overline{\mathcal{S}_1}$ be two cross intersecting families of sets such that $|\mathcal{A}_0| = \binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}$ and $|\mathcal{X}| > \binom{n-3}{r-3} + r^2\binom{n-4}{r-4}$. Then there are elements $u, v \in [n]$ such that every member of $\mathcal{X}$ contains both u and v.*

Proof of Claim 2. Let $\mathcal{A}'_0 = \{D \setminus \{1\} : D \in \mathcal{A}_0\}$, so $|\mathcal{A}'_0| = |\mathcal{A}_0|$. First suppose $\mathcal{A}'_0$ contains a matching $\mathcal{M}$ of size 4. Let $C \in \mathcal{X}$. Since $1 \notin C$, C must intersect each of the members of $\mathcal{M}$. Since $\mathcal{M}$ is a matching, C intersects the 4 members of $\mathcal{M}$ in 4 distinct vertices. Thus the number of such C is at most $(r-1)^4\binom{n-4}{r-4} < |\mathcal{X}|$ for large n, a contradiction. Thus any maximum matching $\mathcal{M}$ of $\mathcal{A}'_0$ satisfies $|\mathcal{M}| \le 3$, so that $|V(\mathcal{M})| \le 3(r-1) < 3r$. Note that $V(\mathcal{M})$ forms a vertex cover of $\mathcal{A}'_0$; or else we can find a larger matching in $\mathcal{A}'_0$. Hence some element u in $V(\mathcal{M})$ lies in at least $|\mathcal{A}'_0|/3r > 3r^2\binom{n-3}{r-3}/3r > r\binom{n-3}{r-3}$ members of $\mathcal{A}'_0$. Let $\mathcal{A}'_0(u)$ be the set of members of $\mathcal{A}'_0$ that contain u, so $|\mathcal{A}'_0(u)| > r\binom{n-3}{r-3}$. Now let $\mathcal{A}''_0(u) = \{E - u : E \in \mathcal{A}'_0(u)\}$, so that $|\mathcal{A}''_0(u)| = |\mathcal{A}'_0(u)| > r\binom{n-3}{r-3}$. Now $\mathcal{A}''_0(u) \subset \binom{[n]-1-u}{r-2}$ so we may apply Lemma 3.4 to see that $\mathcal{A}''_0(u)$ contains a matching (of $(r-2)$-sets) of size $r+1$. Call its members $D_1, D_2, \dots, D_{r+1}$. Let $E_i = D_i \cup \{u\} \in \mathcal{A}'_0$, $1 \le i \le r+1$. Note that u lies in at most $\binom{n-2}{r-2}$ members of $\mathcal{A}'_0$. So there are at least $|\mathcal{A}'_0| - \binom{n-2}{r-2} = 3r^2\binom{n-3}{r-3}$ members of $\mathcal{A}'_0$ not covered by u. These members are covered by $V(\mathcal{M})\setminus\{u\}$. So some element $v \in V(\mathcal{M})\setminus\{u\}$ must lie in at least $3r^2\binom{n-3}{r-3}/(3r-1) \ge r\binom{n-3}{r-3}$ of these members. Hence by Lemma 3.4 as above, there are $(r+1)$ members $F_1, \dots, F_{r+1}$ of $\mathcal{A}'_0$ containing v such that $F_1 \setminus \{v\}, \dots, F_{r+1} \setminus \{v\}$ form a matching in $\binom{[n]-1-v}{r-2}$. Again let $C \in \mathcal{X}$. Since $1 \notin C$ by assumption, C must intersect all members of $\mathcal{A}'_0$; in particular, C intersects all the sets $E_1, \dots, E_{r+1}$. Since $|C| \le r$, it follows that $u \in C$. Similarly, C intersects all the F_i, so $v \in C$. So in order for $\mathcal{X}$ to cross-intersect $\mathcal{A}_0$, all members of $\mathcal{X}$ contain both u and v. ■

For every pair of elements $x, y \in [n]$, let $a(x, y)$ denote the number of sets in $\mathcal{A}$ that contain either x or y or both, and let $b(x, y)$ denote the number of sets in $\mathcal{B}$ that contain either x or y or both. Also let $\mathcal{A}_1(x, y) = \{D \in \binom{[n]}{r} : 1 \le f(D) \le 1 + \max\{\binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}, a(x, y)\} + \binom{n-3}{r-3}\}$.

Claim 3. *Let $\mathcal{X} \subseteq \overline{\mathcal{S}_1}$ satisfy $|\mathcal{X}| > \binom{n-3}{r-3} + r^2\binom{n-4}{r-4}$. Then there exist elements $u, v \in [n]$ such that $\mathcal{A}_1(u, v)$ and $\mathcal{X}$ are not cross-intersecting.*

Proof of Claim 3. Let $\mathcal{A}_0 = \{D \in \binom{n}{r} : 1 \le f(D) \le \binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}\}$ and $\mathcal{A}'_0 = \{D \setminus \{1\} : D \in \mathcal{A}_0\}$, so $|\mathcal{A}'_0| = |\mathcal{A}_0|$. First observe that $\mathcal{A}_0 \subseteq \mathcal{A}_1(u, v)$ for any $u, v \in [n]$. Now suppose that $\mathcal{A}_0$ and $\mathcal{X}$ are not cross intersecting, so that there exist $A \in \mathcal{A}_0$ and $B \in \mathcal{X}$ such that $A \cap B = \emptyset$. Since also $A \in \mathcal{A}_1(u, v)$, we have that $\mathcal{A}_1(u, v)$ and $\mathcal{X}$ are cross intersecting for any $u, v \in [n]$ and we are done. Thus we may assume that $\mathcal{A}_0$ and $\mathcal{X}$ are cross-intersecting. So we can apply Claim 2 to obtain elements $u, v \in [n]$ such that every member of $\mathcal{X}$ contains both u and v.

For this fixed choice of $u, v \in [n]$, we show that $\mathcal{A}_1(u, v)$ and $\mathcal{X}$ are not cross-intersecting. Suppose not. Let $\mathcal{A}'_1(u, v) = \{D \setminus \{1\} : D \in \mathcal{A}_1(u, v)\}$, so $\mathcal{A}'_1(u, v) \subset \binom{[n]-1}{r-1}$. Since $|\mathcal{A}'_1(u, v)| = |\mathcal{A}_1(u, v)| > a(u, v) + \binom{n-3}{r-3}$, there are more than $\binom{n-3}{r-3}$ members of $\mathcal{A}'_1(u, v)$ that contain neither u nor v. Applying Theorem 3.2 with $n-1$ and $r-1$ playing the roles of n and r respectively, and with $t = 2$, we see that among these members there are two, call them E and E', such that $|E \cap E'| \le 1$. First suppose that $E \cap E' = \emptyset$. For any $C \in \mathcal{X}$, C must contain both u and v and intersect each of E and E'. This yields $|\mathcal{X}| \le (r-1)^2\binom{n-4}{r-4}$, contradicting our assumption about $|\mathcal{X}|$. Hence we may assume that $E \cap E' = \{w\}$ for some $w \notin \{1, u, v\}$. Now all members of $\mathcal{X}$ must contain u and v and also intersect E and E'. Thus among them there are at most $\binom{n-3}{r-3}$ that also contain w, and at most $(r-2)^2\binom{n-4}{r-4}$ that do not contain w. Hence $|\mathcal{X}| \le \binom{n-3}{r-3} + (r-2)^2\binom{n-4}{r-4}$, contradicting our assumption about $|\mathcal{X}|$. ■

Symmetric with $\mathcal{A}_1(x, y)$ defined above, let $\mathcal{B}_1(x, y) = \{D \in \binom{[n]}{r} : \binom{n}{r} - f(D) \le \max\{\binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}, b(x, y)\} + \binom{n-3}{r-3}\}$. By a similar argument which we omit, we have the following.

Claim 4. *Let $\mathcal{X} \subseteq \overline{\mathcal{S}_1}$ satisfy $|\mathcal{X}| > \binom{n-3}{r-3} + r^2\binom{n-4}{r-4}$. Then there exist elements $u', v' \in [n]$ such that $\mathcal{B}_1(u', v')$ and $\mathcal{X}$ are not cross-intersecting.*

Now, let $\mathcal{C}$ be the subcollection of $\binom{[n]}{r}$ of minimum size such that $f(\mathcal{C})$ is an interval immediately following $f(\mathcal{A})$, and $|\mathcal{C} \cap \overline{\mathcal{S}_1}| = 1 + \binom{n-3}{r-3} + r^2\binom{n-4}{r-4}$. Similarly, let $\mathcal{D}$ be the subcollection of $\binom{[n]}{r}$ of minimum size such that $f(\mathcal{D})$ is an interval immediately preceding $f(\mathcal{B})$, and $|\mathcal{D} \cap \overline{\mathcal{S}_1}| = 1 + \binom{n-3}{r-3} + r^2\binom{n-4}{r-4}$. When n is sufficiently large, $\mathcal{C}$ and $\mathcal{D}$ are well defined and are disjoint. By definition,

$$|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}| \le \binom{n-1}{r-1} + 2\binom{n-3}{r-3} + 2r^2\binom{n-4}{r-4} + 2. \tag{2}$$

By Claim 3 applied to $\mathcal{D} \cap \overline{\mathcal{S}_1}$ in place of $\mathcal{X}$, there exist elements $u, v \ne 1$ such that some member $D \in \mathcal{D}$ is disjoint from an r-set E satisfying $f(E) \le 1 + \max\{\binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}, a(u, v)\} + \binom{n-3}{r-3}$. Note also that $f(D) \ge \binom{n}{r} - |\mathcal{B}| - |\mathcal{D}|$.

Letting

$$\ell = \binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}$$

we then have

$$dilation(f) \ge |f(D) - f(E)| \ge \binom{n}{r} - |\mathcal{B}| - |\mathcal{D}| - \max\{\ell, a(u, v)\} - \binom{n-3}{r-3}. \tag{3}$$

By a similar argument, for some elements $u', v' \ne 1$, we have

$$dilation(f) \ge \binom{n}{r} - |\mathcal{A}| - |\mathcal{C}| - \max\{\ell, b(u', v')\} - \binom{n-3}{r-3}. \tag{4}$$

Let

$$\lambda_1 = \frac{1}{2}(|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}|), \quad \text{and} \quad \lambda_2 = \frac{1}{2}(\max\{\ell, a(u, v)\} + \max\{\ell, b(u', v')\}).$$

By averaging (3) and (4), we get

$$\mathit{dilation}(f) \geq \binom{n}{r} - \lambda_1 - \lambda_2 - \binom{n-3}{r-3}. \tag{5}$$

By (2),

$$\lambda_1 \leq \frac{1}{2}\binom{n-1}{r-1} + \binom{n-3}{r-3} + r^2\binom{n-4}{r-4} + 1. \tag{6}$$

Letting $m(r,n) = \frac{1}{2}[\binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-2} + \binom{n-5}{r-2}]$, we show that $\lambda_2 \leq m(r,n)$. Observe first that $\lambda_2 = \max\{\frac{1}{2}(\ell + a(u,v)), \frac{1}{2}(\ell + b(u',v')), \frac{1}{2}(a(u,v) + b(u',v')), \ell\}$, and we can bound the expressions in the braces as follows. Certainly $a(u,v)$ is no more than the total number of r-subsets of $[n]$ that contain 1 and at least one of u, v. Thus $a(u,v) \leq \binom{n-1}{r-1} - \binom{n-3}{r-1} = \binom{n-2}{r-2} + \binom{n-3}{r-2}$, and similarly for $b(u',v')$. Thus $\frac{1}{2}(\ell + a(u,v)) \leq \frac{1}{2}[2\binom{n-2}{r-2} + \binom{n-3}{r-2} + 3r^2\binom{n-3}{r-3}] < m(r,n)$ for large n. The same bound holds for $\frac{1}{2}(\ell + b(u',v'))$ symmetrically. Now $a(u,v) + b(u',v')$ is no more than the total number of r-subsets of $[n]$ that contain 1 and at least one of u, v, u', v'. Thus $a(u,v) + b(u',v') \leq \binom{n-1}{r-1} - \binom{n-5}{r-1} = \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-2} + \binom{n-5}{r-2} = 2m(r,n)$, so $m(r,n) \geq \frac{1}{2}[a(u,v) + b(u',v')]$. Finally, since $\ell = \binom{n-2}{r-2} + 3r^2\binom{n-3}{r-3}$, for sufficiently large n we have $m(r,n) > \frac{1}{2}[\binom{n-2}{r-2} + \binom{n-3}{r-2} + \ell] \geq \ell$. So we have shown that $\lambda_2 \leq m(r,n)$.

The required lower bound on *dilation*(f) is now obtained as follows. Applying Lemma 3.5 to the four binomial terms in $m(r,n)$, we get

$$\lambda_2 \leq m(r,n) \leq 2\frac{n^{r-2}}{(r-2)!} - \frac{r+4}{(r-3)!}n^{r-3} + 8r^4n^{r-4}. \tag{7}$$

Hence by (5)–(7), we have

$$\begin{aligned}
\mathit{dilation}(f) &\geq \binom{n}{r} - \lambda_1 - \lambda_2 - \binom{n-3}{r-3}. \\
&\geq \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\binom{n-3}{r-3} - 2\frac{n^{r-2}}{(r-2)!} + \frac{r+4}{(r-3)!}n^{r-3} - 9r^4n^{r-4} \\
&\geq \binom{n}{r} - \frac{1}{2}\binom{n-1}{r-1} - 2\frac{n^{r-2}}{(r-2)!} + \frac{r+2}{(r-3)!}n^{r-3} - 9r^4n^{r-4}. \quad \blacksquare
\end{aligned}$$

References

[1] Peter Allen, Graham Brightwell, Jozef Skokan, Ramsey-goodness—and otherwise, Combinatorica 33 (2) (2013) 125–160.
[2] S.F. Assmann, G.W. Peck, M.M. Sysł o, J. Zak, The bandwidth of caterpillars with hairs of length 1 and 2, SIAM J. Algebr. Discrete Methods 2 (4) (1981) 387–393.
[3] I. Bárány, A short proof of Kneser's conjecture, J. Combin. Theory Ser. A 25 (3) (1978) 325–326.
[4] B. Bollobás, D.E. Daykin, P. Erdös, Sets of independent edges of a hypergraph, Quart. J. Math. Oxford Ser. (2) 27 (105) (1976) 25–32.
[5] Julia Böttcher, Klaas P. Pruessmann, Anusch Taraz, Andreas Würfl, Bandwidth, expansion, treewidth, separators and universality for bounded-degree graphs, European J. Combin. 31 (5) (2010) 1217–1227.
[6] Julia Böttcher, Mathias Schacht, Anusch Taraz, Proof of the bandwidth conjecture of Bollobás and Komlós, Math. Ann. 343 (1) (2009) 175–205.
[7] Ya-Chen Chen, Triangle-free Hamiltonian Kneser graphs, J. Combin. Theory Ser. B 89 (1) (2003) 1–16.
[8] F.R.K. Chung, Labelings of graphs, in: Selected Topics in Graph Theory, vol. 3, Academic Press, San Diego, CA, 1988, pp. 151–168.
[9] J. Chvátalová, On the Bandwidth Problem for Graphs, (Ph.D. thesis), University of Waterloo, 1980.
[10] E. Cuthill, J. McKee, Reducing the bandwidth of sparse symmetric matrices, in: Proc. 24th Natl Conf. ACM, 1969, pp. 151–172.
[11] C. Daubey, U. Feige, W. Unger, Hardness results for approximating bandwidth, J. Comput. System Sci. 77 (2010) 111–132.
[12] J. Diaz, J. Petit, M. Serna, A survey of graph layout problems, ACM Comput. Surv. 34 (3) (2002) 313–356.
[13] V.L. Dolnikov, Transversals of families of sets in $\mathbf{R}^n$ and a relationship between Helly and Borsuk theorems, Mat. Sb. 184 (5) (1993) 111–132.
[14] P. Erdös, A problem on independent r-tuples, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 8 (1965) 93–95.
[15] P. Erdös, Chao Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 12 (1961) 313–320.
[16] U. Feige, K. Talwar, Approximating the bandwidth of caterpillars, Algorithmica 55 (1) (2009) 190–204.
[17] P. Frankl, The Erdos-Ko-Rado theorem is true for $n = ckt$, in: Combinatorial Proc. Fifth Hungarian Coll. Combinatorics, 1978, pp. 365–375.
[18] Peter Frankl, Improved bounds for Erdös' matching conjecture, J. Combin. Theory Ser. A 120 (5) (2013) 1068–1072.
[19] Peter Frankl, Tomasz Łuczak, Katarzyna Mieczkowska, On matchings in hypergraphs, Electron. J. Combin. 19 (2) (2012).
[20] Peter Frankl, Vojtech Rödl, Andrzej Ruciński, On the maximum number of edges in a triple system not containing a disjoint family of a given size, Combin. Probab. Comput. 21 (1–2) (2012) 141–148.
[21] Peter Frankl, Norihide Tokushige, Some best possible inequalities concerning cross-intersecting families, J. Combin. Theory Ser. A 61 (1) (1992) 87–97.
[22] Zoltán Füredi, Cross-intersecting families of finite sets, J. Combin. Theory Ser. A 72 (2) (1995) 332–339.
[23] M.R. Garey, R.L. Graham, D.S. Johnson, D.E. Knuth, Complexity results for bandwidth minimization, SIAM J. Appl. Math. 34 (3) (1978) 477–495.
[24] Norman E. Gibbs, William G. Poole Jr, Paul K. Stockmeyer, An algorithm for reducing the bandwidth and profile of a sparse matrix, SIAM J. Numer. Anal. 13 (2) (1976) 236–250.
[25] Joshua E. Greene, A new short proof of Kneser's conjecture, Amer. Math. Monthly 109 (10) (2002) 918–920.
[26] L.H. Harper, Optimal numberings and isoperimetric problems on graphs, J. Combinatorial Theory 1 (1966) 385–393.
[27] Christoph Helmberg, Franz Rendl, Bojan Mohar, Svatopluk Poljak, A spectral approach to bandwidth and separator problems in graphs, Linear Multilinear Algebra 39 (1–2) (1995) 73–90.

[28] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) 18 (1967) 369–384.
[29] Hao Huang, Po-Shen Loh, Benny Sudakov, The size of a hypergraph and its matching number, Combin. Probab. Comput. 21 (3) (2012) 442–450.
[30] Tao Jiang, Manley Perkel, Dan Pritikin, Arrangements of k-sets with intersection constraints, European J. Combin. 33 (8) (2012) 1882–1899.
[31] Martin Juvan, Bojan Mohar, Laplace eigenvalues and bandwidth-type invariants of graphs, J. Graph Theory 17 (3) (1993) 393–407.
[32] M. Karpinski, J. Wirtgen, A. Zelikovsky, An approximation algorithm for bandwidth on dense graphs, Electron. Colloquium Comput. Complexity 4 (17) (1997).
[33] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Combin. Theory Ser. A 25 (3) (1978) 319–324.
[34] László Lovász, On the Shannon capacity of a graph, IEEE Trans. Inform. Theory 25 (1) (1979) 1–7.
[35] F.S. Makedon, C.H. Papadimitriou, I.H. Sudborough, Topological bandwidth, SIAM J. Algebr. Discrete Methods 6 (3) (1985) 418–444.
[36] Jiři Matoušek, A combinatorial proof of Kneser's conjecture, Combinatorica 24 (1) (2004) 163–170.
[37] Z. Miller, The bandwidth of caterpillar graphs, in: Proceedings of the 12'th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, vol. 33, 1981, pp. 235–252.
[38] Z. Miller, A linear algorithm for topological bandwidth in degree 3 trees, SIAM J. Comput. 17 (5) (1988) 1018–1035.
[39] R. Mohammadyari, M.R. Darafsheh, Topological indices of the kneser graph, Filomat 26 (4) (2012) 665–672.
[40] Burkhard Monien, The bandwidth minimization problem for caterpillars with hair length 3 is np-complete, SIAM J. Algebr. Discrete Methods 7 (4) (1986) 505–512.
[41] G.O. Mota, G.N. Sárközy, M. Schacht, A. Taraz, Ramsey numbers for bipartite graphs with small bandwidth, European J. Combin. 48 (2015) 165–176.
[42] Ch.H. Papadimitriou, The NP-completeness of the bandwidth minimization problem, Computing 16 (3) (1976) 263–270.
[43] Y. Saad, Iterative Methods for Sparse Linear Systems, PWS Publishing Company, 1996.
[44] Horst D. Simon, Shang-Hua Teng, How good is recursive bisection? SIAM J. Sci. Comput. 18 (5) (1997) 1436–1445.
[45] Mario Valencia-Pabon, Juan-Carlos Vera, On the diameter of Kneser graphs, Discrete Math. 305 (1–3) (2005) 383–385.
[46] E.R. van Dam, R. Sotirov, On bounding the bandwidth of graphs with symmetry, INFORMS J. Comput. 27 (1) (2015) 75–88.
[47] D.B. West, Introduction To Graph Theory, second ed., Prentice Hall, 2001.
[48] Richard M. Wilson, The exact bound in the Erdös-Ko-Rado theorem, Combinatorica 4 (2–3) (1984) 247–257.