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Total Closure in Outerplanar Graphs

ARTHUR M. HOBBS
TEXAS A& M UNIVERSITY, COLLEGE STATION, TEXAS

ZEVI MILLER
MIAMI UNIVERSITY, OXFORD, OHIO

Abstract. Given a set A of edges of a graph G, the closure σA of A is the set

$$A \cup \{x \in E(G) : A \cup \{x\} \text{ includes all of the edges of a cycle through } x\},$$

and the dual closure $\sigma^* A$ of A is the set

$$A \cup \{x \in E(G) : A \cup \{x\} \text{ includes all of the edges of an edge cut through } x\}.$$

These are the usual closure and dual closure in matroid theory.

We define the total closure of A, denoted by ΣA, as $\sigma\sigma^*\sigma\sigma^* \cdots \sigma\sigma^* A$ with the closures alternating as long as edges are being added to the set. In this paper we answer the generally difficult question "What is the minimum size of a set A of edges of a graph G if we require $\Sigma A = E(G)$?" for the class of outerplanar graphs. Our solution consists of an algorithm which has polynomial complexity in the number of vertices of the outerplanar graph. This is the first significant class of graphs for which this question has been answered.

Introduction

We use the terminology of Bondy and Murty [**3**] and Welsh [**5**]. We say that edges joining the same two vertices are *parallel*; if an edge e is parallel to other edges, we say e is *in a set of multiple edges*. The *length* of a path is the number of edges in the path. Given a subgraph H of a graph G, the graph $G - H$ is formed from $G - E(H)$ by erasing all vertices of H whose degree is zero in $G - E(H)$.

A set $S \subseteq E(G)$ is called an *edge cut* of G if the number of components of $G - S$ is greater than the number of components of G. An edge cut S will be called a *bond* (following [**3**]) if S is minimal with respect to set inclusion. In particular, if the graph G is connected, then every edge of a bond S joins two different components of $G - S$ and $G - S$ has exactly two components.

Throughout this paper, G will denote a connected outerplanar graph embedded in the plane in such a way that all of its vertices are on the boundary of the unbounded face. We let G^* denote the geometric dual of G.

Given a set A of edges of a graph G, the *closure* σA of A is the set

$$A \cup \{x \in E(G) : A \cup \{x\} \text{ includes all of the edges of a cycle through } x\},$$

and the *dual closure* $\sigma^* A$ of A is the set

$$A \cup \{x \in E(G) : A \cup \{x\} \text{ includes all of the edges of an edge cut through } x\}.$$

In matroid theory, as described in [5], a *closure function* over a finite set S is a function $\mathrm{cl} : 2^S \to 2^S$ satisfying the following four conditions:

(1) For every $X \subseteq S$, we have $X \subseteq \mathrm{cl}X$.
(2) For all $X, Y \subseteq S$, if $X \subseteq Y$, then $\mathrm{cl}X \subseteq \mathrm{cl}Y$.
(3) For every $X \subseteq S$, we have $\mathrm{cl}(\mathrm{cl}X) = \mathrm{cl}X$.
(4) For every $X \subseteq S$ and for every $y, z \in S$, if $y \in \mathrm{cl}(X \cup \{z\}) \setminus \mathrm{cl}X$, then $z \in \mathrm{cl}(X \cup \{y\}) \setminus \mathrm{cl}X$.

If G is a graph, then the function σ defined above is precisely the closure function of a matroid on the edge set $E(G)$ of G; this matroid is called the *cycle matroid* of the graph.

A planar graph G has a dual G^*, and each edge cut of G^* is dual to a cycle in G. It is not difficult to show [5] that the cycle matroid of G^* is isomorphic to a matroid on $E(G)$ defined by using σ^* for the closure function.

It is not usual in matroid theory to apply both σ and σ^* to subsets in the same matroid. However, in [4], Hobbs and Vince observed that such a combination could be used in network design to allow frequent testing of a complex circuit with a minimum of disturbance of the circuit. In an electrical circuit in which all elements are described in terms of resistance as a function of time t, we can use Ohm's Law $E(t) = R(t)I(t)$ to convert between voltage and current values in various elements. Hence the Kirkhoff current and voltage laws can be applied to circuits and cut sets, respectively, to determine voltage drops and currents when all but one element of a circuit or all but one element of a cut set have known voltage drops or currents. A minimum disturbance of a circuit in testing will occur if a smallest possible set of voltage drops and currents are measured to supply enough information to calculate all of the others. Thus we want to find a minimum set E of elements of the electrical circuit such that applying the Kirkhoff laws and Ohm's Law repeatedly will give voltage drop and current values on every element of the network.

In a graph G, the corresponding objective is to find a minimum set A of edges such that repeatedly applying σ and σ^* will generate all of $E(G)$. Therefore, we define the *total closure* of A, denoted by ΣA, as $\sigma\sigma^*\sigma\sigma^* \cdots \sigma\sigma^* A$ with the closures alternating as long as edges are being added to the set. The question "What is the minimum number of closure operations σ and σ^* needed to get from A to $\Sigma A = E(G)$?" was addressed by Akkari and Hobbs in [2]. Thus, in this paper we feel free to always use σ^* as the first (right-most) closure applied to A. However, since $\sigma\sigma S = \sigma S$ and $\sigma^*\sigma^* S = \sigma^* S$ for any set S of edges of G, the last (left-most) closure may be either σ or σ^*. In this paper we let Σ stand both for the function converting A into ΣA and for the sequence of σ's and σ^*'s, and we allow the context to distinguish between the two uses.

An example is shown in Figure 1. There, the edges are picked up in order $1, 2, 3, \cdots$, starting with set A consisting of the three edges marked with a. On the edge labels, a prime means the edge was picked up by σ and a star means the edge was picked up by σ^*. Note that the horizontal edge marked with both "x" and "5^*" is picked up by σ^* using the edge cut indicated by the edges marked with "x". This shows that, to be picked up by σ^*, it is not necessary for an edge to be adjacent to edges already picked up.

A set $A \subseteq E(G)$ is a *starter set for* G if $\Sigma A = E(G)$. The electrical circuit problem becomes here, "What is the minimum cardinality $\alpha(G)$ of a starter set for graph G?" Let a starter set $A \subseteq E(G)$ with $|A| = \alpha(G)$ be called a *minimum starter set*. In an earlier investigation [4], it was shown that determining $\alpha(G)$ is a difficult problem because minimal starter sets are not necessarily minimum starter sets. In this paper for the first time we provide the solution of the problem of determining $\alpha(G)$ for a significant class of graphs.

The Structure of Outerplanar Graphs

In [4], Hobbs and Vince proved that, if a graph H has blocks $B_1, B_2, \cdots, B_k$, then $\alpha(H) = \Sigma_{i=1}^{k} \alpha(B_i)$. Thus we may restrict our attention to the problem of finding minimum starter sets in two-connected graphs. Further, there is a routine proof from the definitions that if $A, B \subseteq E(G)$, then $\Sigma(A \cup \Sigma B) = \Sigma(A \cup B)$. Thus we can add edges one–by–one to a starter set without worrying about the order in which we select the edges; we can concentrate on being sure that we have the elements of a minimum set. Hobbs and Vince also showed that, if $B \subseteq \Sigma A$, then $\Sigma B \subseteq \Sigma A$. Thus, if we have occasion to use a set B in a starter set, if $|A| \leq |B|$, and if $B \subseteq \Sigma A$, then we are not worse off if we substitute A for B. Finally, if G is a cycle, then $\alpha(G) = 1$. Thus we shall consider only two-connected outerplanar graphs that are not cycles.

The edges of an outerplanar graph may be classified as *outer* if they are on the boundary of the infinite face and *inner* if they are not on that boundary.

A two-connected outerplanar graph G with at least three vertices is Hamiltonian, and the boundary of the outer face is a Hamiltonian cycle. Further, if e is any inner edge of such a graph, either portion of the boundary Hamiltonian cycle cut off by e together with e and all edges interior to the resulting cycle constitute a two-connected outerplanar subgraph of G; let us call these two subgraphs the *lobes with respect to* e. If F is a bounded face of G and e is an inner edge of G on the boundary of F, exactly one lobe L with respect to e fails to include F; call that lobe *the* e*-lobe of* F, and call L a *lobe at* F.

We will occasionally view a lobe as a graph independent of the graph G. In that case, the edge e is the only inner edge of G which is an outer edge in each of its lobes; all other edges remain outer or inner as they are in G. Also, a lobe of a two-connected outerplanar graph G, when viewed as a graph independent of G, is also a two-connected outerplanar graph and is therefore Hamiltonian.

Suppose L is the e-lobe of face F, and suppose $E(L)$ does not consist entirely of edges parallel to e. Then L includes a unique bounded face F_L which has on its boundary both an edge not parallel to e and either e or an edge parallel to e. We say that F_L is the *key face* of L.

Of course, a lobe consisting of a set of multiple edges cannot have a key face. But trivially, if L is such an e-lobe, then $E(L) \subseteq \Sigma A$ if and only if $e \in \Sigma A$. Thus such lobes can be dealt with while we are studying more complicated lobes.

Let w be the vertex of the dual G^* in the infinite face of a two-connected outerplanar graph G. It is easy to see that if we erase w from G^*, we get a tree T, which we call the *inner dual tree* of G. The *inner dual tree* of a lobe L of face F in G is that part of the inner dual tree T of G that is induced by the vertices of T corresponding to the faces of L. If G is not a cycle, each vertex of degree one of T corresponds to a face in G which has exactly one inner edge on its boundary. Let us call such a face an *end face*. If G is a cycle, we take its only bounded face to be an *end face*. It is clear that every lobe in G includes one or more end faces.

The concepts of the preceding paragraphs are illustrated in Figure 2. Here we see the inner dual tree T shown by dashed edges with its vertices numbered. The e-lobe L has the faces corresponding to vertices 1 through 9 of T. Note that e-lobe L of face F has key face K and end faces D_1 and D_2. The inner dual tree of lobe L is the subtree of T induced by the vertices labeled 1 through 9.

LEMMA 1. *Let F be a bounded face of G, and let C be a cycle of G having F in its interior. Let e be an inner edge of G on the boundary of F and let L be the e-lobe of F. Then $E(L) \cap E(C) \neq \emptyset$. Further, if a cycle C' of G includes both edges of L and edges not in L, then F is in the interior of C'.*

PROOF: Suppose F is in the interior of C and $E(C) \cap E(L) = \emptyset$. Erase all of the edges of L from G. The resulting graph has F as part of the unbounded face and C is present in that graph. Hence F is not interior to C, contrary to supposition. Next, suppose C' includes both edges in L and edges not in L; then it must include the ends of e. Let P be the path in C' joining these two vertices and including no edges of L. Then P and e together form a cycle C'' of G which includes the boundary edge e of F. Let L' be the other e-lobe of G; then C'' is contained in $L' = G - L + e$ and includes e as an outer edge of L' which is also a boundary edge of F. Hence erasing e from L' both destroys C'' and combines F with the infinite face, so F is interior to C''. But the interior of C'' is interior to C'. The lemma follows. ∎

LEMMA 2. *Let G be a two-connected outerplanar graph, and let e be an inner edge of G. Let L and L' be the e-lobes of G.*

(1) *If S is a bond of G and if $S \cap E(L) \neq \emptyset$, then $|S \cap E(L)| \geq 2$.*
(2) *If S' is a bond of G and if $e \in S'$, then S' includes outer edges of both $L - e$ and $L' - e$.*
(3) *If S'' is a bond of G and if S'' includes edges of both $L - e$ and $L' - e$, then $e \in S''$.*

PROOF: For the first part, since L is Hamiltonian, if $S \cap E(L) = \{f\}$, then the ends of f are in the same component of $G - S$. Hence, S is not a bond, contrary to supposition. The first part of the lemma follows.

For the second part, because the boundary of the infinite face of G is a Hamiltonian cycle in G, the ends of e are joined by a path P_L in $L - e$ and by a path $P_{L'}$ in $L' - e$, and these paths consist solely of outer edges of G. Since S' is a bond, the ends of e must be separated by S', so S' includes edges of both P_L and $P_{L'}$, as claimed.

For the third part, let X be the set of end vertices of e. If $e \notin S''$, then the vertices of X are in one component Q of $G - S''$. Since S'' is a bond and includes an edge of L, there is a vertex v of L not in Q. But every path in G from v to a vertex not in L includes a vertex of X; hence $S'' \cap E(L)$ includes an edge cut of G. Since S'' is a bond, $S'' \subseteq E(L)$, contrary to supposition. The result follows. ∎

Let G be an outerplanar graph, and let $A \subseteq E(G)$. Suppose L is an e-lobe of face F. We *condense* L by forming the graph $G - L + e$. Given set $A \subseteq E(G)$, we form the *A-condensed graph* $G'(A)$ by condensing each lobe L which is maximal under subgraph inclusion with respect to the condition $E(L) \subseteq \Sigma A$. When no confusion can arise, we use G' for $G'(A)$ and we call G' the *condensed graph* of G. Note that G is the condensed graph

$G'(\emptyset)$. The set of base edges of maximal lobes of G which were condensed in forming G' together with the edges in $A \cap E(G')$ is denoted by A'; we say the elements of A' are *A-marked*, and we will say that A' is a *starter set* for G'. We will use the notations G' and A' throughout the rest of this paper with meanings given them in this paragraph. Note that $E(G') \setminus \Sigma A' = E(G) \setminus \Sigma A$. Further, if $A'' \subseteq E(G')$ with $A' \subseteq A''$, then $E(G') \setminus \Sigma A'' = E(G) \setminus \Sigma(A \cup (A'' \setminus A'))$. Thus to explore the effect of adding edges to A in G, it suffices to add edges to A' in G'. Finally, note that, if L' is a lobe of G', then $E(L') \setminus \Sigma A' \neq \emptyset$ by the definition of G'.

Let L be an e-lobe which has a vertex other than the ends of e, and let F_L be the key face of L. We say L is *open* if there are no edges in parallel with e and an A–marked outer edge of G' is on the boundary of F_L. If a lobe is not open, it is *closed*. Thus a lobe with with at least one edge in parallel with the base edge is necessarily closed, as is any lobe whose key face has no A–marked edge on its boundary.

Let G be a two-connected outerplanar graph, and let A be a subset of $E(G)$. For the two-connected condensed outerplanar graph $G' = G'(A)$ with starter set A', the set of A–marked edges of G', we give the following recursive definition of a tube. Notice in this definition that tubes are inductively lobes, and so terminology applicable to lobes, such as "open," is applicable to tubes.

(1) An e-lobe L whose inner dual tree is a path is a *tube*; the boundary of its end face is *the tip of* L, and its *base* is the edge e.

(2) Let L be an e-lobe of G' and suppose an open tube L_1 with base f is a subgraph of L. Form an e-lobe L' from L by condensing L_1. We say that L is a *tube* if L' is a tube. The *tip* of L is the tip of L', and the *base* of L is e.

If a tube L is an e-lobe, we shall refer to L as an *e-tube*. If the inner dual tree of a tube is a path, then the tube is called *simple*. The *length* of a simple tube is the number of bounded faces in the tube. If G' has an edge e such that both e-lobes of G' are tubes, then we say *G' is a tube*; the *tip* of G' is the tip of either of the e-lobes, arbitrarily chosen.

These concepts are illustrated in Figure 3. As shown in Figure 3(a), initially G has five tubes L_1, L_2, L_3, L_4, and L_5, all simple. The tips of L_1 through L_5 are marked D_1 through D_5, respectively, and the base edges of L_1 through L_5 are marked e_1 through e_5, respectively. Note that tube L_4 has only one face, its key face; the tip of L_4 is the boundary of its only face.

After selecting edge a_1 for set A, condensing L_1, and A–marking edge e_1 as A_1, we get the condensed graph $G'(\{a_1\})$ shown in Figure 3(b). Here the tip D_2 of L_2 has become the tip D_6 of the larger tube L_6 with base edge e_6. Note that choosing a starter edge in D_2 instead of D_1 would lead to the same situation of an A–marked edge on the boundary of face F_1 as part of an e_6 tube of face F_3.

We then select edge a_2 for the second edge in A. Now tube L_4 is condensed and an open tube L_7 with key face F_2 having A–marked edge A_2 on its boundary is formed; this is shown in Figure 3(c). Applying step (2) of the definition of a tube, the e_6-lobe containing F_3 is a tube because L_7 is open. Since the e_6-lobe containing F_1 is also a tube, $G'(\{a_1, a_2\})$ is a tube. Either D_6 or D_3 could be taken as the tip of $G'(\{a_1, a_2\})$. We chose D_3 for this figure. Note that selecting a starter edge in D_5 instead of choosing edge a_2 in L_4 would have resulted in the same situation of an A–marked edge on the boundary of key face F_2

of an e_7-tube which is open.

THEOREM 1. *Let G be a two-connected outerplanar graph with inner dual tree T. Suppose T has $k \geq 2$ end vertices. Then $\alpha(G) \leq k-1$.*

PROOF: If T is a path, G is a simple tube and $\alpha(G) = 1$. Thus we may suppose for induction that $k \geq 3$ and that the theorem is true for outerplanar graphs having inner dual trees with at most $k-1$ end vertices.

Let f be an outer edge of the boundary of an end face of G, and let L be the longest simple tube of G which includes as a tip the boundary of the end face containing f; let e stand for the base of L. Since $k \geq 3$, the edge e is a boundary edge of a face F of G corresponding to a vertex of degree three or more of T. Let $A = \{f\}$; it is easily seen that $E(L) \subseteq \Sigma f$. Then we may condense L and A-mark e. The resulting graph G' is two-connected, outerplanar and has an inner dual tree with $k-1$ end vertices. Hence $\alpha(G') \leq k-2$. Letting A' be a set of $\alpha(G')$ edges of G' such that $\Sigma A' = E(G')$, since $e \in \Sigma f$ it follows that $\Sigma(A' \cup \{f\}) = E(G)$. ∎

Now we will study the interaction between a tube and the rest of the graph under total closure. Our plan in this paper is to let L be a tube of G', let A'' be a subset of $E(G')$ such that $A' \subseteq A''$ and $E(L) \cap A'' = E(L) \cap A'$, and investigate under what conditions $E(L) \subseteq \Sigma A''$ or $E(L) \subseteq \Sigma(A'' \cup \{f\})$ where f is an edge of the tip of L. Following the lemmas dealing with these questions, we provide an algorithm which uses these results to determine the exact value of $\alpha(G)$ and to find a minimum starter set for any two-connected outerplanar graph G.

In accordance with our plan, for the rest of the paper, given condensed graph G' with A-marked edge set A', and given a lobe L of G', we will let $A''(L) = A' \cup B$ where $B \subseteq E(G') \setminus E(L)$. When no confusion is possible, we denote $A''(L)$ by A''. Exactly which edges are in B will not matter to us; the only purpose of B is to have $|\Sigma A''| > |\Sigma A'|$.

In what follows, given a subgraph H of a graph M, and given a set B of edges of M, we will have occasion to use that part Σ' of the sequence Σ from its first application to $B \setminus E(H)$ to the last closure in Σ that does not bring in an edge of H. If $[\Sigma(B \setminus E(H))] \cap E(H) = \emptyset$, then we let Σ' be Σ. We will describe Σ' as *the longest H-avoiding closure sequence from B*. If H has just one edge e, we will say Σ' is *the longest e-avoiding closure sequence from B.*

Suppose e is an edge of $G' - A'$ such that the longest e-avoiding closure sequence Σ' from A'' is not Σ. Then either $\sigma\Sigma' A'$ or $\sigma^*\Sigma' A'$ includes e; let us say *e is picked up by σ* or σ^*, respectively. For any cycle C in $\Sigma' A' \cup \{e\}$ such that $e \in E(C)$, we say e is picked up by σ *using cycle C*. Similarly, if S is a bond of G' and if $S \subseteq \Sigma' A' \cup \{e\}$, and if $e \in S$, then we say that e is picked up by σ^* *using S*.

Given a lobe L with base e of G', suppose $A'' = A' \cup B$, where $B \subseteq E(G') \setminus E(L)$. We let $A_L = A' \cap E(L) = A'' \cap E(L)$ and $\overline{A_L} = A'' \setminus E(L)$. We now explore the interactions between A_L and $\overline{A_L}$ in the total closure, and we will show that $\Sigma_1 A_L$ and $\Sigma_2 \overline{A_L}$ interact only through the base edge of L, where Σ_1 is the longest $(G' - L + e)$-avoiding closure sequence from A'', and Σ_2 is the longest L-avoiding closure sequence from A''.

LEMMA 3. *Let F be a bounded face of G', and let L be an e-lobe of F. Further, let $A'' = A' \cup B$ with $B \subseteq E(G') \setminus E(L)$, and let Σ' be the longest L-avoiding closure sequence* *from A''. Suppose $E(L)\setminus\Sigma A_L$ is a proper subset of $E(L)\setminus\Sigma A''$. Then $e\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$ or $e\in\sigma^*(\Sigma'\overline{A_L}\cup\Sigma A_L)$.*

PROOF: Under the hypotheses, there is an $f\in E(L)\setminus\Sigma A_L$ such that $f\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$ or $f\in\sigma^*(\Sigma'\overline{A_L}\cup\Sigma A_L)$. If $f\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$, we note that there cannot be a path in ΣA_L joining the ends of f, for otherwise f would be in ΣA_L. But a cycle in $\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$ consisting of edges of both L and $L'=G'-L$ must include a path in $\Sigma'\overline{A_L}$ joining the ends of e since L and L' share only e and its ends. It follows that $e\in\sigma\Sigma'\overline{A_L}$. Next suppose $f\in\sigma^*(\Sigma'\overline{A_L}\cup\Sigma A_L)$. Since $f\notin\Sigma A_L$, in this case f is picked up by σ^* using an edge cut S of G' which includes f and one or more edges of $\Sigma'\overline{A_L}$. Thus by Lemma 2, e is in S. Thus, $e\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$ or $e\in\sigma^*(\Sigma'\overline{A_L}\cup\Sigma A_L)$. ∎

Let F be a bounded face of G', and let L be an e-lobe of F. Further, let $A''=A'\cup B$ with $B\subseteq(E(G')\setminus E(L))$, and let Σ' be the longest L-avoiding closure sequence from A''. Suppose $E(L)\setminus\Sigma A_L\neq E(L)\setminus\Sigma A''$. We say L is *entered at e by σ* if $e\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$, and we say L is *entered at e by σ^** if $e\in\sigma^*(\Sigma'\overline{A_L}\cup\Sigma A_L)$.

Given a cycle C of an outerplanar graph M, a *slice to the interior of* C is a minimal set of edges S of M such that a point of the interior of C is a part of the unbounded face of $M-S$. If F is a bounded face of M, a *slice to* F is a slice to the interior of the boundary of F. A slice to F or to the interior of C *avoids* a subgraph H if it includes none of the edges of H. We use a slice by adding a few edges to it, thereby producing an edge cut of G'. Picturing the graph as drawn on a sheet of paper, we visualize a slice as a cutting of the paper from the edge of the paper through edges of the graph until we reach a point in the interior of the cycle. In Figure 4, the edge set $\{e_1,e_2,e_3\}$ is a slice to the interior of the cycle C marked by bold edges. This set of edges is also a slice to face F.

LEMMA 4. *Suppose every edge in A' is an outer edge of G'. Let L be an e-lobe of bounded face F, and suppose that L is entered at e by σ. Let Σ' be the longest L-avoiding closure sequence from $\overline{A_L}$. Then $\sigma\Sigma'\overline{A_L}$ includes the edges of a slice to F which avoids L.*

PROOF: If any edge cut of G' including edges of F is a subset of $\Sigma'\overline{A_L}$, the result is immediate, so we may suppose that no such edge is picked up by σ^*. Thus we may suppose all edges of F that are included in $\Sigma'\overline{A_L}$ are in $\overline{A_L}$ or were picked up by σ. Let C be a cycle including e and otherwise including only edges of $\Sigma'\overline{A_L}$. C exists since $e\in\sigma(\Sigma'\overline{A_L}\cup\Sigma A_L)$. Since we are assuming in this paper that Σ begins with σ^*, the sequence Σ' is not the empty sequence. Further, since L is entered by σ, the last (left-most) closure in Σ' is σ^*. Let Σ'' be the sequence Σ' with the last (i.e., left-most) σ^* deleted.

We now show that $\Sigma'\overline{A_L}$ includes a slice to the interior of C. If Σ'' is the empty sequence, then the edges of C other than e are in $\overline{A_L}$. Since these edges are outer edges of G', any one of them constitutes a slice S' to the interior of C. But if Σ'' is not the empty sequence, and all of the edges of $C-e$ were in $\Sigma''\overline{A_L}$, then so would e be in that set since Σ'' ends with a σ. Hence, some edge f of $C-e$ is in an edge cut $S\subseteq\Sigma'\overline{A_L}$. Thus again $\Sigma'\overline{A_L}$ includes a slice $S'\subseteq S$ to the interior of C.

Since G' is outerplanar, every edge interior to C joins two vertices of C and so is included in $\sigma\Sigma'\overline{A_L}$. Thus in both cases, the union of S' with $E(C)$ and the interior edges of C includes a slice to F which avoids L. ∎

We need to verify that certain properties of a simple tube are properties of all tubes.

LEMMA 5. *Suppose a tube L of G' is entered by σ^* from $A'' = A' \cup B$ with $B \subseteq E(G') \setminus E(L)$. Then $E(L) \subseteq \Sigma A''$. Further, all outer edges of L other than the A–marked edges and edges in sets of multiple edges are picked up by σ^*.*

PROOF: For a simple tube, this is easily checked. If L is a tube formed by k iterations of step (2) of the definition of a tube, if the lemma is true for tubes requiring at most $k-1$ steps, and if L_1 with base g is the last open tube to be added in step (2) in the formation of L, then g is outer in $L - L_1 + g$. By induction, $E(L - L_1 + g) \subseteq \Sigma A''$. Further, g is not in a set of multiple edges since L_1 is open. Thus g is picked up by σ^* in $G' - L_1 + g$ by the induction hypothesis. But the edge cut in $G' - L_1 + g$ which picks up g extends to an edge cut in G' which picks up g since L_1 is open and g is on the boundary of the key face of L_1. Thus $E(L_1) \subseteq \Sigma A''$ by the induction hypothesis, so $E(L) \subseteq \Sigma A''$ and all outer edges of the tube that are neither A–marked nor in sets of multiple edges are picked up by σ^*. ∎

LEMMA 6. *Suppose every edge in A' is an outer edge of G'. Let tube L be an e-lobe of bounded face F of G'. Let $A'' = A' \cup B$ with $B \subseteq E(G') \setminus E(L)$. Suppose both that $e \in \Sigma\overline{A_L}$ and that L is entered by σ. Then $E(L) \subseteq \Sigma A''$ and all outer edges of L other than A–marked edges and edges in multiple sets of edges are picked up by σ^*.*

PROOF: If L is a simple tube and if one or more edges parallel to e exist in G', then all of the edges parallel to e are in $\Sigma\overline{A_L}$. Thus the lemma is true if L has no vertex other than the ends of e. By Lemma 4, $\Sigma\overline{A_L}$ includes a slice to F which avoids L. Suppose L has key face F_L, and let edge f be an outer edge of G' on the boundary of F_L. Then the slice to F extends to an edge cut in G' by including the edge e, all edges parallel to e, and the edge f. As noted before, since L is simple, from here it is easy to see that all of L will be included in $\Sigma\overline{A_L}$ and all outer edges of L other than A–marked edges and edges in multiple sets of edges are picked up by σ^*.

But suppose L is constructible by $k \geq 1$ iterations of step (2) of the definition of a tube, and suppose the lemma is true for all tubes constructible in $k-1$ or fewer iterations of step (2). Let L_1 be the last tube added in the construction of L, and suppose L_1 has base g. By definition, L_1 is open and has an A–marked edge h such that g and h are on the boundary of its key face. By the induction hypothesis, since g is not in a set of multiple edges, g is picked up by σ^* in $G' - L_1 + g$. Let Σ' by a longest g-avoiding closure sequence. Since L_1 is open, the edge cut in $G' - L_1 + g$ extends to an edge cut in G' with the additional edge h. Since h is A–marked, this edge cut has only the edge g not in $\Sigma' A''$, so $g \in \Sigma A''$. Thus L_1 is entered by σ^*, and so by Lemma 5 the lemma holds for L_1. Hence the lemma holds for L. ∎

LEMMA 7. *Let L be a tube of G' with tip D, and let f be an outer edge of L in D. Then $f \notin A'$ and $E(L) \subseteq \Sigma(A' \cup \{f\})$. Further, every outer edge of L which is neither A–marked nor in a set of multiple edges not in A' is picked up by σ^* in $\Sigma(A' \cup \{f\})$.*

PROOF: This is easy if L is a simple tube. In particular, since every lobe of G' has at least one edge not in $\Sigma A'$, we have $f \notin A'$. But suppose L is constructible by $k \geq 1$ iterations of step (2) of the definition of a tube, and suppose the lemma is true for all tubes constructible in $k-1$ or fewer iterations of step (2). Let L_1 be the last tube added in the construction of L, and suppose L_1 has base g. Then D is not in L_1, L_1 is open, and g is not in a set of multiple edges. Since g is an outer edge of $L - L_1 + g$, it is picked up by σ^* from $(A' \cup \{f\}) \cap E(G' - L_1 + g)$ by the induction hypothesis. But L_1 is open, so L_1 is entered by σ^* as in Lemma 6. Hence $E(L_1) \subseteq \Sigma(A' \cup \{f\})$ by Lemma 5, and every outer edge of L_1 which is neither in A' nor in a set of multiple edges is picked up by σ^*. But by induction the lemma is true for $E(L - L_1 + g)$; thus the lemma holds for L. ∎

LEMMA 8. *If every edge in A' is outer in $G'(A)$, then no inner edge of a tube of G' can be in $\Sigma A'$.*

PROOF: Suppose, to the contrary, that inner edge f of tube L is in $\Sigma A'$.

Note that one of the f-lobes of G' is contained within L; this lobe L_1 is also a tube of G'. If L_1 contains the tip of L, then the tip D of L_1 is that tip. If L_1 does not contain the tip of L, then the boundary of some end face of L_1 is the tip of a sub-tube of a tube L' used in the recursive construction of the tube L. In this case, let D be this tip. It suffices to show that D is in $\Sigma A'$, for then all of L_1 is in $\Sigma A'$ by Lemma 7; in that case L_1 will have been condensed in forming G', contrary to the assumption that L_1 is a tube of G'.

First, suppose that L_1 is simple. Let P be the inner dual tree of L_1, and let f' be the inner edge of L_1 in $\Sigma A'$ which is closest to D measured along P. If $f' \in E(D)$, the result is immediate, so suppose $f' \notin E(D)$. Let L_2 be the f'-lobe containing D, and let F_{L_2} be the key face of L_2, if it exists. Suppose f' is picked up by σ using cycle C. Since any cycle including f' in L_2 must include either an outer edge of D or an inner edge of L_2, none of which are in $\Sigma A'$, C cannot be included in L_2. Hence $f' \in \Sigma A_{\overline{L_2}}$ and so $E(L_2) \subseteq \Sigma A'$ by Lemma 6, a contradiction. But now suppose f' is picked up by σ^* using bond S. Then f' is not in a set of multiple edges, the key face F_{L_2} of L_2 exists, and S includes another edge g of the boundary of F_{L_2} by Lemma 2. By the choice of f' and the assumption that L_2 is simple, g is an A-marked outer edge of G'. It is now easy to verify that $E(L_2) \subseteq \Sigma A'$, a contradiction again.

But suppose L_1 is constructible by $k \geq 1$ iterations of step (2) of the definition of a tube, and suppose the lemma is true for all tubes constructible in $k-1$ or fewer iterations of step (2). Choose f' and L_2 as before, but use for the path P the path of the inner dual tree of G' which extends to D from a face incident with f. By the choice of f', none of the inner edges of L_2 closer to D along P than f' is in $\Sigma A'$, so a cycle including f' and otherwise in $\Sigma A'$ is in $G' - L_2 + f'$ as before. Thus a contradiction follows from Lemma 6 if f' is picked up by σ. Suppose f' is picked up by σ^* using bond S of G'. Then L_2 is entered by σ^*, so by Lemma 5 with B the empty set, $E(L_2) \subseteq \Sigma A'$. This contradicts the definitions of G' and A'. The lemma follows. ∎

LEMMA 9. *Suppose every edge in A' is an outer edge of G', and suppose tube L is an e-lobe of some bounded face of G'. Let $A'' = A' \cup B$ with $B \subseteq E(G') \setminus E(L)$. If L is closed, then L cannot be entered by σ^*.*

PROOF: Since L has a Hamiltonian cycle all of whose edges except e are outer edges of G', in order for L to be entered by σ^*, ΣA_L must include a slice to the key face F_L of L. Since L is closed, this slice must include an inner edge f of G' other than e on the boundary of F_L. But the existence of f contradicts Lemma 8. ∎

The effect of the preceding lemmas is to show that tubes are isolated from one another except through their bases. Given a two-connected outerplanar graph G and starter set A, let T' be the inner dual tree of $G' = G'(A)$. If G' is not a tube, in G' each maximal tube L has an inner dual tree T_L. For each maximal tube L of G', erase $V(T_L)$ from T'. The result is a subtree U of T'; we call U the ***indicator tree*** of G'.

In Figure 2, a graph G is shown with inner dual tree T indicated by dashed edges. Here $A = \emptyset$, so all tubes are simple. There are five simple tubes here; these have the sets of faces $\{1,2\}, \{4,5,6,7\}, \{11,12\}, \{14\}$, and $\{15,16\}$. Note that the indicator tree U for this graph is the subtree of T induced by the vertices labeled $3, 8, 9, 10$, and 13. If the lobe containing the faces numbered 1 and 2 is condensed, the new graph has the same simple tubes as before except that the tubes having face sets $\{1,2\}$ and $\{4,5,6,7\}$ are replaced by a simple tube having face set $\{7,6,5,4,3,8,9\}$. Thus the inner dual tree loses vetices 1 and 2 and the indicator tree becomes the subtree induced by vertices 10 and 13 only. When the additional lobe containing face 14 is condensed, the lobe containing faces 13, 15, and 16 is an open tube and the indicator tree becomes vertex 10 only.

The next lemma will be used in the algorithm which follows it.

LEMMA 10. *Let G be a two-connected outerplanar graph and let A be a subset of $E(G)$. Let $G' = G'(A)$ be the A-condensed graph of G. Suppose G' is not a tube. Let G' have inner dual tree T' and let U be the indicator tree of G'. Then either U has only one vertex, and every lobe of the corresponding face is a tube, at least three of them closed, or U has a vertex of degree one. In this second case, for each vertex v of degree one in U, and letting F_v be the face of G' corresponding to v, at least two of the lobes of F_v are closed tubes.*

PROOF: Since G is not a tube, U has a vertex. If U has just one vertex, every lobe of the face F of G' corresponding to that vertex is a maximal tube of G' by the definition of U. But if two or fewer of those tubes were closed, then G' would be a tube by definition, contrary to assumption. Thus at least three of the lobes of the face corresponsing to the vertex of U are closed tubes in this case.

Alternatively, U has vertices of degree one. But each vertex v of degree one of U corresponds to a face F_v of G' such that all but one of the lobes in G' of F_v is a maximal tube. If at most one of these tubes were closed, by the definition of a tube, F_v would be part of a larger tube in G' incorporating all of the tubes which are lobes of F_v, contradicting the maximality of the tubes used in defining U. Thus each vertex of degree one of U corresponds to a face of G' having at least two closed tubes as lobes. ∎

The Algorithm

We next provide an algorithm which will find a minimum starter set for any two–connected outerplanar graph, thus completely solving the minimum starter set problem for this important class of graphs. Let G be an two–connected outerplanar graph, and let A be a subset of the edges of G. Let $H = G'(A)$. For use in this algorithm and the following work, we define *Opface(H)* as the set of faces of H corresponding to the end vertices of the indicator tree of $H = G'(A)$. In this definition, if the indicator tree has just one vertex, we take that vertex to be the end vertex of the tree.

ALGORITHM

(Input: Outerplanar graph G. Output: A minimum starter set A for G.)

If G is a simple tube, let f be an outer edge of a tip of G and let $A = \{f\}$. Then $\Sigma A = E(G)$ and $\alpha(G) = 1$. Stop.

Otherwise,

(Initialization) $H = G$ and $A = \emptyset$.

1. For each element $F \in Opface(H)$, do
   begin
   $A = A \cup S$, where S consists of one outer edge from the tip of each of the closed tubes of F except one.
   end
2. $H =$ the A-condensed graph of G.

If H is a tube, then
  $A = A \cup \{f\}$, where f is an outer edge of the tip of G'.
  $\Sigma A = E(G)$ and $\alpha(G) = |A|$. Stop.

Otherwise, go to line 1.

As presented here, this algorithm does not describe how to make the choices of tubes and edges in the tips of those tubes that will be used for the sets S. Such a choice can be imposed by, for example, requiring that the edges be numbered and the choice be the first numbered edge possible at each decision. However, we prefer not to be so definite here, since other choice rules may be preferable on occasion. For example, in the next two theorems, we have need of the rule that edges from the boundary of a prespecified face X shall not be selected by the algorithm.

Figures 5 and 6 illustrate this algorithm. In Figure 5, since the graph $H = G$ is not a tube, the set $Opface$ is formed, containing the two faces P_1 and P_2 shown in Figure 5(a). A choice of tubes of each of P_1 and P_2 is made and the edges marked a_1 and a_2 are selected for the set S. Lobes L_1 and L_4 are condensed, resulting in the graph $H = G'(\{a_1, a_2\})$ shown in Figure 5(b). Since the lobe L_7 is open, this graph is a tube. The algorithm may choose between an edge in tip D_3 or an edge in the tip D_6. We selected edge a_7 in tip D_3 for the last edge of A. In Figure 6, we have numbered the edges as they are picked up by the selection and subsequent closure processes. Here, edges selected for the starter set are marked with "a" and a subscript placing them in the ordering of edges, while edges picked up by σ^* are marked with a star and edges picked up by σ are marked by a prime. We see that $\alpha(G) = 3$.

Verification

THEOREM 2. *Let G be a two-connected outerplanar graph, and suppose G is not a cycle. If G has a minimum starter set contained in the set R of outer edges of G, then the algorithm finds a minimum starter set for G which is contained in R. Further, if X is any bounded face of G, the choices in the algorithm can be made so that the minimum starter set includes no edge of the boundary of X.*

PROOF: First note that the set of edges selected by the algorithm is contained in R. Thus it suffices to show that the set found is a minimum set and that it avoids X if X has been specified.

Since $\Sigma\emptyset$ is the set of one-edge edge cuts and loops of G, and since G is two-connected, and so has no one-edge edge cuts or loops, we conclude that $\alpha(G) > 0$. If G is a simple tube, the algorithm selects an outer edge f from the boundary of an end face of G. Since $\Sigma\{f\} = E(G)$, $\alpha(G) = 1$ and the algorithm is verified in this case. Further, the selection of f can be made to avoid the boundary of X, so the theorem holds in the case that G is a simple tube.

Suppose M_0 is a starter set for G, and suppose $M_0 \subseteq R$. Suppose further that G is not a simple tube, so the indicator tree U_0 of G has one or more vertices. It will suffice to show that M_0 can be replaced by a starter set M which the algorithm can find and such that $|M| \leq |M_0|$. Further, M_0 could be a minimum starter set, so it follows that M is a minimum starter set.

To show that the step between lines 1 and 2 of the algorithm is well–defined, recall that $Opface(G)$ is the set of faces of G corresponding to vertices of degree at most one in U_0. By Lemma 10, for each face $F \in Opface(G)$, two or more lobes of F are closed simple tubes. Thus the set S of that step is non–empty.

We will now show that M_0 shares edges with all but one simple tube of F, and this is true for each face in $Opface(G)$. For suppose two of the closed simple tubes of F, e.g. L_1 and L_2, share no edges with M_0. Let Σ' be the longest $(L_1 \cup L_2)$-avoiding closure sequence. Supposing there is a slice in $\Sigma' M_0$ to F, since L_1 and L_2 are closed, they cannot be entered by σ^* (Lemma 9). But any cycle having only one edge not in $\Sigma' M_0$ and that one edge in L_1 must have F in its interior (Lemma 1), and so it must include edges of both L_1 and L_2. Since $E(L_1 \cup L_2) \cap \Sigma' M_0 = \emptyset$, no such cycle exists. Similarly, there is no cycle having only one edge not in $\Sigma' M_0$ and that one edge in L_2. Thus $E(L_1 \cup L_2) \cap \Sigma M_0 = \emptyset$, contrary to the choice of M_0.

Let k be the number of passes through statements 1 through 2 of the algorithm.

Suppose $k = 1$. The algorithm selects one edge of the tip of each of all but one of the simple tubes of F for each $F \in Opface(G)$; because of the free choices in the algorithm, we may require that the selected edges be from tips of tubes containing edges of M_0. Let A_1 be the set of all edges so selected as we scan through $Opface(G)$, let G_1 be the condensed graph $G'(A_1)$, and form set M_1 as the disjoint union $A_1 \cup (M_0 \cap E(G_1))$. By the choices of A_1 and M_1 and by Lemma 7, every edge of M_0 that has been deleted in forming M_1 is included in ΣA_1, so $\Sigma M_1 = E(G)$. But by construction, $|M_1| \leq |M_0|$. Note that the union of the set of A–marked edges of G_1 with $M_1 \setminus A_1$ is a starter set for G_1, and note that $M_1 \setminus A_1 \neq \emptyset$ by the definition of G_1. Since $k = 1$, G_1 is a tube, so the algorithm will select an edge f of the boundary of one end face of G_1. Then $\Sigma(A_1 \cup \{f\}) = E(G)$ and $|A_1 \cup \{f\}| \leq |M_0|$. Thus the algorithm finds a minimum size starter set for G in this case.

Assume $k > 1$ and suppose we have found G_i as the condensed graph $G'(A_1 \cup \cdots \cup A_i)$ where $A_j, j = 1, 2, \cdots, i$ is the set of edges selected by the algorithm in pass j through lines 1 through 2. Suppose further that M_i is the disjoint union $A_1 \cup \cdots \cup A_i \cup (M_0 \cap E(G_i))$; then $\Sigma M_i = E(G)$. Suppose that the union of all A–marked edges of G_i together with the edges in $M_0 \cap E(G_i)$ is a starter set for G_i. Finally, suppose $M_0 \cap E(G_i) \neq \emptyset$. Note that $A_1 \cup \cdots \cup A_i$ is the set A and G_i is the graph $G'(A)$ formed after i passes through lines 1 through 2 of the algorithm.

When $i < k$, then G_i is not a tube. Let U_i be the indicator tree of G_i and (applying

Lemma 10) recall that, for every face $F \in Opface(G_i)$, F has at least two lobes L_1 and L_2 which are closed tubes. If $M_i \cap E(L_1 \cup L_2) = \emptyset$, then $\Sigma M_i \cap E(L_1 \cup L_2) = \emptyset$ as before. Hence M_i fails to have edges in at most one of the closed tubes of each face in $Opface(G_i)$. The algorithm selects one outer edge of the tip of all but one of the closed tubes of F for each $F \in Opface(G_i)$. Because of the free choices in the algorithm, we may require that the selected edges be from tips of tubes containing edges of M_i. Let the selected edges constitute a set A_{i+1}. Let $G_{i+1} = G'(A_1 \cup \cdots \cup A_{i+1})$. Form M_{i+1} as the disjoint union $A_1 \cup \cdots \cup A_{i+1} \cup (M_0 \cap E(G_{i+1}))$. Note that the union of the A-marked edges of G_{i+1} with the edges in $M_{i+1} \setminus (A_1 \cup \cdots \cup A_{i+1})$ is a starter set for G_{i+1}. Finally, note that $M_{i+1} \setminus (A_1 \cup \cdots \cup A_{i+1}) \neq \emptyset$ by the definition of G_{i+1}. Now repeat the preceding paragraph and this paragraph with i replaced with $i+1$.

When $i = k$, then G_i is a tube, and the algorithm will select an edge f of the boundary of one end face of G_i. Then $\Sigma(A_1 \cup \cdots \cup A_i \cup \{f\}) = E(G)$ and $|A_1 \cup \cdots \cup A_i \cup \{f\}| \leq |M_0|$. Thus the algorithm finds a minimum size starter set for G if the algorithm stops after k passes through lines 1 through 2.

Since the algorithm stops just after the end of pass k through lines 1 through 2 when G_k is a tube, we see that the algorithm forms a starter set for G which is no larger than M_0. Thus the algorithm forms a minimum starter set for G if there is a minimum starter set which consists entirely of outer edges of G.

Further, any starter set formed by making one set of the choices called for in the algorithm is the same size as a starter set formed by making a different set of choices in the algorithm. Thus at every step in the algorithm we may choose not to select a prespecified end face. It follows that if X is any bounded face of G and if G is not a cycle, the choices in the algorithm can be made to find a minimum starter set consisting of outer edges of end faces of G such that none of the edges is on the boundary of X. ∎

THEOREM 3. *Let G be a two-connected outerplanar graph. Then there is a minimum starter set for G containing only outer edges of G.*

PROOF: If G is a cycle, the theorem is trivial. Suppose G is a smallest outerplanar graph for which every minimum starter set for G includes an inner edge, and let M be a minimum starter set for G with as few inner edges as possible. Since the inner edges of G are all crossed by edges of the inner dual tree, the tree structure assures that there is an inner edge E in M such that one e-lobe L of G has no other inner edges of G in M. Let L' be the other e-lobe of G. Suppose L' is a simple tube. Let f be an outer edge in the tip of L'. Since E is an element of $\Sigma\{f\}$, we can replace e in M with f and have a starter set of the same size with fewer inner edges, contrary to the choice of M. Thus L' is not a simple tube.

Form an outerplanar graph H from L by adding an edge e' parallel to e. Since L' is not a simple tube, H is smaller than G. Embed H in the plane so that the digon formed by e and e' bounds a face X of H, e is an outer edge of H, and every edge of L that is outer in G is outer in H.

Next we show that $M' = M \cap E(L)$ is a starter set for H. Let g be in $E(L) \setminus \{e\}$ and suppose Σ' is a longest g-avoiding closure sequence in G from M. Suppose g is picked up by σ, and let C be a cycle containing g such that the edges of C other than g are in $\Sigma' M$. If C includes any edges of $G - L$, then C includes a path P in $G - L$ joining the ends of e;

replacing P in C by e produces a cycle C' contained in L such that the edges of C' other than g are contained in $(\Sigma' M) \cap E(L)$.

If g is picked up by σ^*, and if an edge cut S used to pick up g includes edges of $G - L$, then S includes e by Lemma 2. Hence $S' = S \setminus E(G - L)$ is an edge cut of L (viewed as an independent graph) such that $S' \setminus \{g\}$ is contained in $(\Sigma' M) \cap E(L)$.

In both cases, it follows that g is in $\Sigma(M \cap E(L))$. Hence $M' = M \cap E(L)$ is a starter set for L. Since e' is in $\sigma\{e\}$, we thus have that M' is a starter set for H.

Since H is smaller than G, H has a minimum starter set contained in the outer edges of H. (This starter set might conceivably be smaller than M', which is in the outer edges of H.) Hence, by Theorem 2, the algorithm can be used to find a minimum starter set F for H which is contained in the outer edges of H and which includes neither of the edges on the boundary of X. Since M' is a starter set for H, $|F| \leq |M'|$.

Also, since e and e' are parallel in H, they are picked up by σ from F. It follows that $E(H)$ is contained in ΣF even in the graph $G - L + H = G + e'$. But F is contained in $E(L)$, so $E(L)$ is contained in ΣF in G. Since e is in $E(L)$, and is thus in ΣF, it follows that $F \cup (M \setminus E(L)) = M''$ is a starter set for G. Since $|F| \leq |M'| = |M \cap E(L)|$, we have that $|F \cup (M - E(L))| \leq |M|$. Thus $F \cup (M - E(L))$ is a minimum starter set of G. But $F \cup (M - E(L))$ has fewer inner edges of G than M has, contary to the choice of M. The theorem follows. ∎

COROLLARY. *Let G be a two-connected outerplanar graph. Then the algorithm finds a minimum starter set for G consisting entirely of outer edges of G. Further, if X is any bounded face of G, the choices in the algorithm can be made so that the minimum starter set includes no edge of the boundary of X.*

PROOF: By Theorem 3, we can add the condition that there is a minimum starter set consisting entirely of outer edges of G. But then this corollary is Theorem 2. ∎

Finally, we have a simple lower bound for $\alpha(G)$ when G is a two-connected outerplanar graph.

THEOREM 4. *Let G be a two-connected outerplanar graph with inner dual tree T. Then $\alpha(G) \geq \max\{1, \Delta(T) - 1\}$.*

PROOF: If T is a path, then G is a simple tube and $\alpha(G) = 1$, so the theorem holds in this case. Let F be a bounded face of G corresponding to a vertex of T of degree $\Delta(T) \geq 3$. Each lobe of F either is a tube or includes a face F' and a lobe of F' which is a tube, and all lobes are closed before any edges are selected for A. Suppose we select a set of outer edges for A, no more than one per lobe of F and fewer than $\Delta(T) - 1$ in total. Let L_1 and L_2 be lobes of F which include no edges of A. Let Σ' be a longest $(L_1 \cup L_2)$-avoiding closure sequence. Even supposing there is a slice in $\Sigma' A$ to F, since L_1 and L_2 are closed, they cannot be entered by σ^* (Lemma 9). But any cycle having only one edge not in $\Sigma' A$ and that one edge in L_1 must have F in its interior (Lemma 1), and so it must include edges of both L_1 and L_2. Since $E(L_1 \cup L_2) \cap A = \emptyset$, no such cycle exists. Similarly, there is no cycle having only one edge not in $\Sigma' A$ and that one edge in L_2. Thus $E(L_1 \cup L_2) \cap \Sigma A = \emptyset$. It follows that $\alpha(G) \geq \Delta(T) - 1$. ∎

Time Complexity

We give a rough sketch of the time complexity of the algorithm, showing that it is of polynomial time as a function the number of vertices in a given two–connected outerplanar graph G.

We will need the following notation. Let p and q be the number of vertices and edges in G respectively. Let $T = T_G$ be the inner dual tree of G. Given a tree S, suppose we traverse its vertices by "walking up" from its end vertices; that is, we traverse it in "postorder" as described in [1], page 54. For any vertex $v \in V(S)$, let $B(v)$ be the subtree of S consisting of v and the vertices lying "below" v in this traversal; namely, v together with the vertices whose postorder number is less than the postorder number of v in this traversal. The vertices in $B(v)$ correspond to faces of G, and the union of those faces and their boundaries form a lobe of the face of G corresponding to the vertex immediately above v in the tree T_G. Denote this lobe by $B_L(v)$.

Recall that G^* denotes the geometric dual of G, and let the vertex of G^* in the unbounded face of G be z. Then T_G is just $G^* - z$. For any subset Q of $E(G)$, let Q' be the subset of $E(G^*)$ consisting of edges which cross the edges in Q; if $Q = \{e\}$, for simplicity we use e' for Q'.

Let us divide the running time of the algorithm into stages, the i'th stage being the i'th passage through statements 1 through 2. (The time thereby left unaccounted for, e.g. the intialization and final update $A = A \cup \{f\}$, is bounded by a constant additive term and hence can be neglected in the asymptotics.) The tasks performed in the i'th stage may be divided into two categories.

1) Finding elements of $Opface(H)$, and
2) Performing condensation to obtain the new $H = G'(A)$.

Consider category 1) in a given stage. The task is to find $Opface(H)$, which amounts to finding the end vertices of the indicator tree of H. It therefore suffices to find the indicator tree itself, from which its end vertices are easily found. This is of course equivalent to finding the maximal tubes of H, and deleting them from H.

A first approximation of how to do this is as follows. Let T_H be the inner dual tree of H. We walk up T_H in postorder, passing information in this traversal as follows. Let $v \in V(T_H)$, and let the children of v in $B(v)$ (these are the vertices immediately "below" v in the traversal) be denoted v_i, for $i \geq 1$. Assume that there resides at each v_i information of whether $B_L(v_i)$ is a tube or not, and if it is whether it is open or closed. Then one can determine the same information for v in accordance with the definition of a tube (a step that requires $O(d_{T_H}(v))$ time, where $d_{T_H}(v)$ is the degree of v in T_H), and this information is placed at v. This process is repeated; that is, we continue the postorder traversal together with the dynamic determination of the information at each new vertex v reached, until a vertex v is reached for which $B_L(v)$ is found not to be a tube. The lobes $B_L(v_i)$ are then maximal tubes. Such a vertex v is a vertex of the indicator tree of H. To find the indicator tree we then delete from T_H the branches $B(v_i)$ of all the children v_i, $i \geq 1$, of v, and we do this at every such v.

To get the best time bound, the above procedure is modified so that at each stage we start the postorder traversal of the current H at end vertices of the indicator tree of the previous H, thereby using already available information without recomputing it. The result is that no vertex of T_H is processed more than once in a given stage, and no vertex of T_G (of which all the successive T_H's are subtrees) is processed in two different stages. It follows that the time spent on category 1), summed over all stages, is $O(|V(T)|\Delta(T))$.

Consider now category 2) in a given stage. Suppose the current H has r edges. Let A'' be the union of set of A–marked edges of H together with the edges added to A in the loop following statement 1 of the algorithm. Let Σ' be an initial (right–hand end) subsequence of the sequence of σ's and σ^*'s used in producing $\Sigma A''$, and let $P = \Sigma' A''$; thus P is A'' if Σ' is empty. Recall that P' is the subset of $E(G^*)$ consisting of edges which cross the edges in P.

Let $e \in E(H) \setminus P$. We wish to determine if e can be picked up by either σ or σ^*. Let $H[P]$ be the subgraph of H induced by P, and assume that we have the components structures of $H[P]$ and $H^*[P']$ in a form in which all vertices of $H[P]$ and $H^*[P']$ have, in addition to their own names, integer labels such that two vertices have the same label iff they are in the same component.

Observe that e is picked up by σ if and only if both ends of e lie in the same component of $H[P]$. The latter condition can be checked in $O(1)$ time, since this involves just noting whether the two ends of e have labels, and if so whether these labels are the same. If it is found that e is indeed picked up by σ, then we update P by $P = P \cup \{e\}$ and P' by $P' = P' \cup \{e'\}$. The components structure of the new $H[P]$ remains the same as that of the old $H[P]$. Since a cycle in H corresponds to an edge cut of H^*, the edge e' joins two different components of $H^*[P']$. We update the components structure of $H^*[P']$ by coalescing the two components joined by e' by relabeling the vertices in these two components with the smaller of the two labels used in the components.

Next observe that e is picked up by σ^* if and only if, for some subset $D \subseteq P$, we have that $D \cup \{e\}$ is an edge cut of G. But $D \cup \{e\}$ is an edge cut of G exactly when $D' \cup \{e'\}$ forms a cycle in G^*. Hence, given e, we are reduce to the following problem:

Determine if there is a subset $D' \subseteq P'$ such that $D' \cup \{e'\}$ is a cycle of G^*.

This problem can be solved by the same method as above, assuming a similar knowledge of the components structure of $H^*[P']$.

If we find that e is picked up by σ^*, then we update P and P' by $P = P \cup \{e\}$ and $P' = P' \cup \{e'\}$, and we further update the components structure of $H[P]$ by coalescing the two components joined by e by relabeling the vertices in these two components with the smaller of the two labels used in the components. The components structure of the new $H^*[P']$ remains the same as that of the old $H^*[P']$. Thus for a given P and P', the time needed for determining if a given $e \in E(H) \setminus P$ is picked up by σ^* is $O(\max\{|V(H^*)|, |V(H)|\})$, including all necessary updates invoving P and P'. For our purposes, we may replace $O(\max\{|V(H^*)|, |V(H)|\})$ by $O(p)$.

For a given P, we examine at most $|E(H)| - |P| = r - |P|$ edges of H before possibly updating P and P'. We have completed the condensation process in step 2 of the algorithm when we have found a P for which no edge of $E(H) \setminus P$ can be picked up. Then the new H will be the old H with the edges of this final P condensed. Thus in a given stage having current graph H on r edges, category 2) requires time at most

$$O(\sum_{i=0}^{r} p(r-i)) = O(r^3).$$

The total time for the algorithm spent on category 2), over all stages is then

$$O(\sum_{i=1}^{s} r_i^3),$$

where s is the number of stages and r_i is the number of edges in the graph H in stage i. Now $s \leq |V(T)|$ since at each stage we get a new, smaller indicator tree, while $r_i \leq q$ for all i. Hence the total time of category 2) is bounded by $O(|V(T)|q^3)$.

Finally the total time for both categories is therefore $O(p + |V(T)|q^3) = O(|V(T)|q^3)$. Now since G is outerplaner, we have $q = O(p)$, while by Euler's formula we have $|V(T)| = O(q-p)$. Thus the total time required by the algorithm is bounded by $O(p^4)$.

An Unsolved Problem

The general problem of determining the value of $\alpha(G)$ for an arbitrary graph G is still unsolved. One particularly interesting example is the grid $G(m,n)$ formed as the Cartesian product $P_m \times P_n$, where P_m and P_n are paths of lengths m and n respectively, and $n \geq m$. If we draw this graph as a grid on the plane with copies of P_m represented by vertical paths and copies of P_n represented as horizontal paths, and if we let A be the set of vertical edges on the left side of the graph, then $\Sigma A = E(G)$. Thus $\alpha(G) \leq m = \min\{m,n\}$. We conjecture that $\alpha(G(m,n)) = m$. This conjecture appears to be quite difficult.

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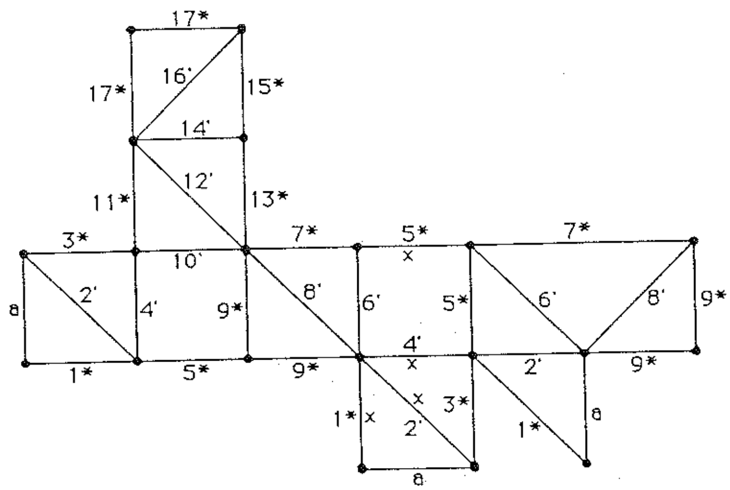


Figure 1: The total closure using the starter set consisting of the three edges maked a

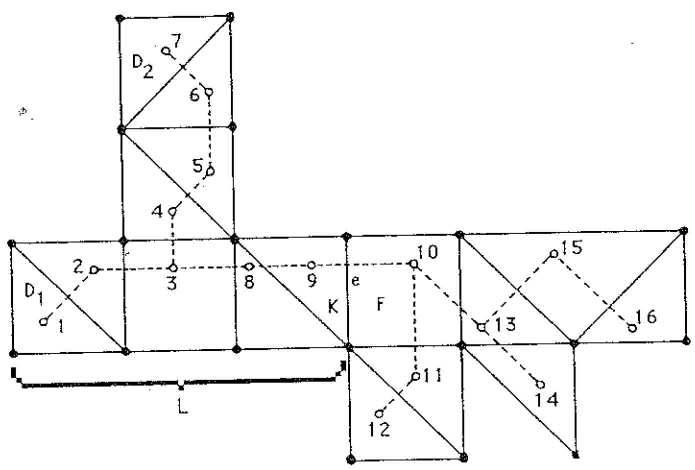


Figure 2: The inner dual tree, and the lobe L having base edge e, key face K, and end faces D_1 and D_2

(a) G

(b) G'({a1})

(c) G'({a1,a2})

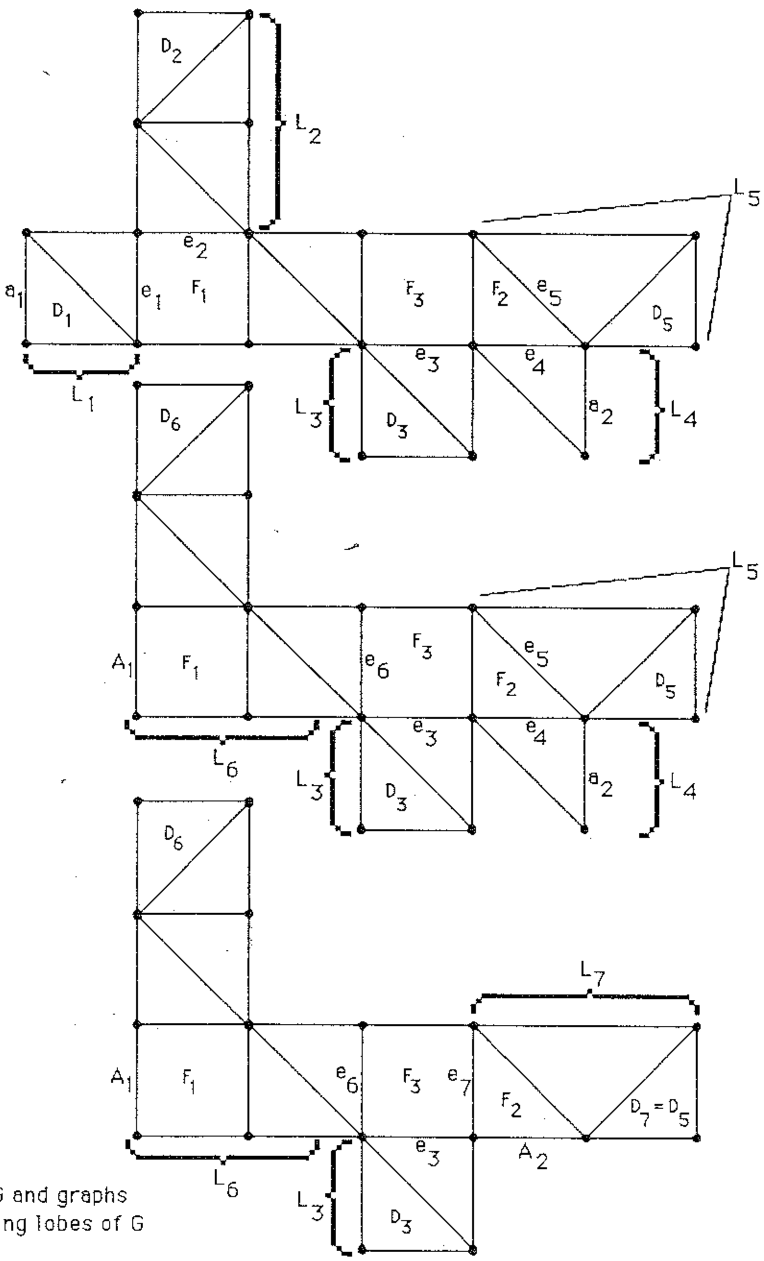


Figure 3: The graph G and graphs obtained by condensing lobes of G

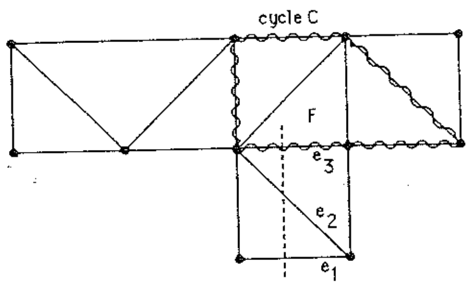


Figure 4: A slice

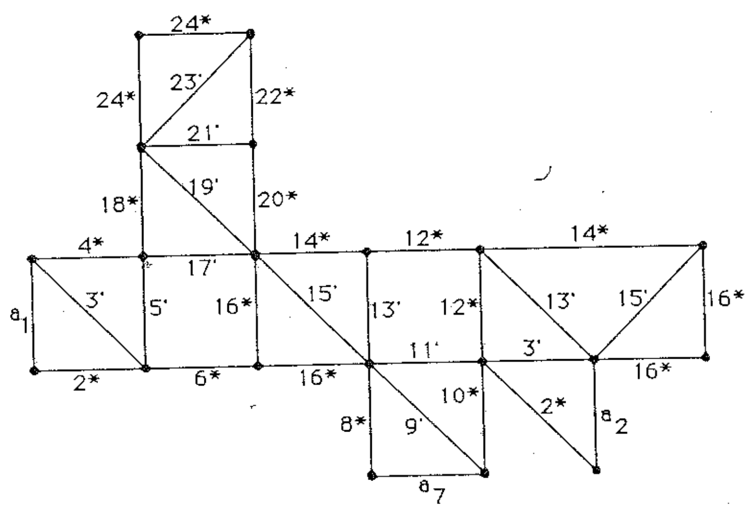
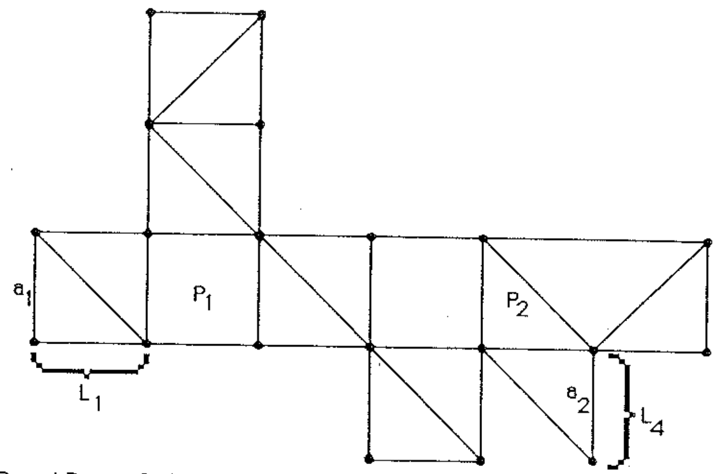
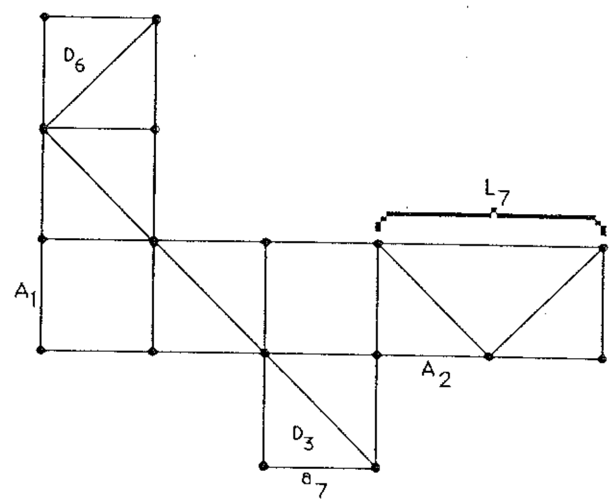


Figure 6: The edges marked with the order in which they are picked up by the algorithm



(a) P_1 and P_2 are Opfaces of $H = G$



(b) $H = G'(\{a_1, a_2\})$

Figure 5: Tubes and Opfaces for G used by the algorithm