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THE BICHROMATICITY OF A LATTICE-GRAPH

FRANK HARARY, DERBIAU HSU and ZEVI MILLER

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Abstract

The bichromaticity $\beta(B)$ of a bipartite graph B has been defined as the maximum order of a complete bipartite graph onto which B is homomorphic. This number was previously determined for trees and even cycles. It is now shown that for a lattice-graph $P_m \times P_n$ the cartesian product of two paths, the bichromaticity is $2+\{mn/2\}$.

The *bichromaticity* of a connected bipartite graph B, written $\beta(B)$, has been defined as the maximum order $p = r + s$ of a complete bigraph $K_{r,s}$ onto which B is homomorphic, no two points of B of different colors being sent to the same point. This is the invariant for bigraphs corresponding to the *achromatic number* of a graph G, defined by Harary and Hedetniemi (1970) as the maximum order of a complete graph onto which G is homomorphic.

The *majority* of B is the color class of maximum cardinality μ in B. It was shown by the present authors (1977) that for a tree, $\beta = 1 + \mu$ and for an even cycle C_{2n}, $\beta = 1 + n$ if n is odd and $2 + n$ if n is even.

The terminology and notation of the book of Harary (1969) will be used. The *lattice-graph* is the cartesian product $P_m \times P_n$ of two paths; it is obviously bipartite. We now develop a formula for its bichromaticity. To do this, we recall a basic but simple lemma proved in our previous paper.

LEMMA. *If $h: B \to K_{r,s}$ is a bicomplete homomorphism of a noncomplete bigraph B onto $K_{r,s}$ then $rs \leqq q$ and $r + s \geqq \mu + 1$.*

THEOREM. *The bichromaticity of a lattice-graph is*

$$\beta(P_m \times P_n) = 2 + \{mn/2\}. \tag{1}$$

PROOF. For the sake of convenience, we first introduce some notation. Let $A = \{mn/2\}$. View $P_m \times P_n$ as a lattice having m rows and n columns, where v_{ij} is the point in row i and column j. As a connected bigraph, $P_m \times P_n$

has the unique color classes C and D, where without loss of generality C contains all points v_{ij} in the same color class as v_{11} and D all v_{ij} in the same color class as v_{12}. Thus $|C| = A = \mu(P_m \times P_n)$ and $|D| = [mn/2]$.

In order to show that the right side of (1) is a lower bound for $\beta(P_m \times P_n)$, we will define a bicomplete homomorphism $f: P_m \times P_n \to K(2, A)$ as follows. Let $X_1 = \{v_{ij}: j \text{ odd}, v_{ij} \in D\}$ and $X_2 = \{v_{ij}: j \text{ even}, v_{ij} \in D\}$, so D may be partitioned $D = X_1 \cup X_2$. To describe the action of f on C, let all points in X_1 be sent by f to one point v, and all points in X_2 to another point w. On the other hand, f leaves all points of C fixed. Then clearly $f(P_m \times P_n) = K(2, \{mn/2\})$, establishing the lower bound.

Our proof that the right side of (1) is also an upper bound for $\beta(P_m \times P_n)$,

$$(2) \qquad \beta(P_m \times P_n) \leqq 2 + A,$$

is considerably more involved. In order to accomplish this, it is sufficient to demonstrate the following proposition:

(P) If $h: P_m \times P_n \to K_{r,s}$ is a homomorphism for which $r + s = \beta(P_m \times P_n)$, then $r \leqq 2$.

Once (P) has been verified, we have established (2) since $s \leqq \mu = A$. We begin by giving a simple argument that the possibilities $r = 3$ and $r = 4$ lead to contradictions. Later we shall see that $r \geqq 5$ is also impossible.

Assume first that $r = 3$. The majority C must contain at least two of the corner points of the lattice; call them v and w. Each such point has degree 2 and its image under h must have degree 3 since $r = 3$. Therefore such a point is identified with at least one other point in C under h. We may then have either v and w identified under h with different points or with the same point. In both cases, the image of C under h will contain two fewer points than C, that is,

$$s \leqq \overline{A - 2}.$$

Thus as a bound for $r + s$, we get

$$r + s \leqq 3 + A - 2 = 1 + A.$$

But this contradicts $r + s = \beta(P_m \times P_n) \geqq 2 + A$, as the lower bound in (1) is already verified.

Assume now that $r = 4$. Since $s \geqq r = 4$, the number of points in the lattice is at least 8. Thus C must contain at least three of the points on the "boundary" of the lattice. Each of these three points has degree at most 3 and the degree of its image under h is 4. Thus as above each of these three points must be identified with other points in C by the homomorphism h. No matter how this is done, the image of C under h will contain at least 3 fewer points than C, so that

$$s \leqq A - 3.$$

Again, we get as a bound

$$r + s \leqq 4 + A - 3 = A + 1,$$

contradicting $\beta(P_m \times P_n) \geqq 2 + A$.

Having completed our consideration of the cases $r = 3$ and 4, we turn to an analysis of the remaining possibility $r \geqq 5$. First we show that any lattice satisfying the hypothesis of the lemma with $r \geqq 5$ has at most 72 points. Using this bound, it will then be shown that no such lattice can exist.

Applying the lemma to lattice-graphs, we have

$$rs \leqq q(P_m \times P_n) = 2mn - m - n. \tag{3}$$

Because we have already established $\beta(P_m \times P_n) \geqq 2 + \{mn/2\}$, we get

$$r + s \geqq A + 2 \geqq \frac{mn}{2} + 2. \tag{4}$$

By combining inequalities (3) and (4), we find

$$\frac{2mn - m - n}{r} + r \geqq A + 2 \geqq \frac{mn}{2} + 2.$$

However, the usual inequality between the arithmetic mean and the geometric mean implies at once that $m + n \geqq \sqrt{2mn}$. This inequality combined with the preceding one immediately gives

$$r + \frac{2mn - \sqrt{2mn}}{r} \geqq \frac{mn}{2} + 2.$$

For convenience, let us write $y = \sqrt{(mn)}$. It then follows directly that

$$2r^2 - 4r \geqq (r-4)y^2 + 2\sqrt{2}y. \tag{5}$$

We now discuss the problem in two cases: $r \geqq 6$ and $r = 5$.

Case 1. $r \geqq 6$. Here (5) gives

$$2r^2 > 2r^2 - 24 \geqq 2r^2 - 4r \geqq y^2 + 2\sqrt{2}y \geqq y^2,$$

so that

$$r > \frac{y}{\sqrt{2}}. \tag{6}$$

Furthermore

$$r \leqq \frac{q(P_m \times P_n)}{s} \leqq \frac{q(P_m \times P_n)}{r}$$

implies

$$r^2 \leqq 2mn - m - n < 2mn = 2y^2.$$

This gives

$$r \leqq \sqrt{2}y$$

Combining this with (9), we find

$$s + \sqrt{2}y \geqq r + s \geqq \frac{mn}{2} + 2 > \frac{y^2}{2}.$$

Then we have

$$(7) \qquad s > \frac{y^2}{2} - \sqrt{2}y.$$

But

$$2y^2 = 2mn > 2mn - m - n \geqq rs > \left(\frac{y^2}{2} - \sqrt{2}y\right)\frac{y}{\sqrt{2}},$$

where the last inequality follows from (6) and (7). Hence we get $\sqrt{mn} = y < 6\sqrt{2}$, so that $mn < 72$.

CASE 2. Suppose $r = 5$. Then (5) gives $y < 6$ so that $mn < 36$.

Now Cases 1 and 2 have shown that under the hypothesis of (P) with $r \geqq 5$, the lattice-graph $P_m \times P_n$ can have at most 72 points. As a first step in showing that such lattices cannot in fact exist, we prove that these lattices having $r \geqq 5$ must have at least 20 points. By the condition that $r \geqq 5$, there exists a homomorphism $h: P_m \times P_n \to K_{r,s}$ with $s \geqq r \geqq 5$ and $\beta(P_m \times P_n) = r + s \geqq 10$. We now view h as a sequence of elementary homomorphisms. Since $p(K_{r,s}) \geqq 10$ and each elementary homomorphism reduces the number of points in the lattice-graph by one, we see that h is composed of a sequence of at most $mn - 10$ elementary homomorphisms. We note further that since each elementary homomorphism fixes all but two points, the maximum number of points in $P_m \times P_n$ not fixed by h is $2(mn - 10)$. Now assume that $mn < 20$ or in other words that $2(mn - 10) < mn$. This means that at least one point v of $P_m \times P_n$ is left fixed by h. Its degree in $h(P_m \times P_n)$ is then at most its degree in $P_m \times P_n$. That is,

$$5 \leqq \text{degree of } h(v) \text{ in } K_{r,s} \leqq \text{degree of } v \text{ in } P_m \times P_n \leqq 4.$$

a contradiction which shows that $mn \geqq 20$.

The next step will be to prove that a lattice satisfying the hypotheses of (P) with $r \geqq 5$ can have at most 16 points. This combined with $mn \geqq 20$ gives the final contradiction to $r \geqq 5$ and proposition (P) will then be proved. Note first that as the maximum degree satisfies $\Delta(P_m \times P_n) \leqq 4$, each point of C must experience at least one identification under h if $r \geqq 5$. In fact, each point of C must have as many identifications as the smallest multiple of 4 greater than or equal to r. Therefore

$$s \leqq A/\{r/4\}.$$

Now the conditions $mn < 72$ and $r \geqq 5$ give $s \leqq 18$ as an upper bound. Furthermore if $r \geqq 13$, then $s \leqq 9$, contradicting $r \leqq s$. We thus conclude $5 \leqq r \leqq 12$.

We now get an upper bound for mn in terms of r by showing $mn \leqq 2(r-2)/(1-1/\{r/4\})$. Suppose this last inequality is false. Then using $r \geqq 5$ and manipulating this inequality routinely, we get

$$r + A/\{r/4\} < 2 + A.$$

But by the above discussion, the left side is at least as large as $r + s$. Therefore

$$r + s < 2 + A.$$

This contradicts $\beta(P_m \times P_n) = r + s$ since the lower bound $\beta(P_m \times P_n) \geqq 2 + A$ has already been established.

As before, the homomorphism h is composed of at most $mn - (r + s)$ elementary homomorphisms. Since $r + s \geqq 2r$, we get

$$mn - (r+s) \leqq 2((r-2)/(1-1/\{r/4\})) - 2r.$$

Thus we have for an upper bound for the number of points not fixed by h,

$$2(mn - (r+s)) \leqq 4((r-2)/(1-1/\{r/4\})) - 4r.$$

The right side assumes the maximum 16 among the integers r satisfying $5 \leqq r \leqq 12$. But this implies that if the lattice has more than 16 points, it has at least one fixed point under h. As observed previously, this contradicts the fact that each point in $h(P_m \times P_n) = K_{r,s}$ has degree at least 5. Therefore we conclude that if $r \geqq 5$, then $mn \leqq 16$, yielding the long awaited contradiction. The statement (P) is now proved and the theorem follows.

Conclusion and unsolved problems

1. For the cylinder $C_{2n} \times P_m$ we have the following partial result.

$$\beta(C_{2n} \times P_m) = \begin{cases} 3 + mn \text{ if } 3 \mid n \text{ or } 3 \nmid n \text{ with } n \text{ even and } m \text{ odd} \\ 2 + mn \text{ or } 3 + mn \text{ otherwise} \end{cases}$$

We conjecture that "otherwise" always yields $\beta = 2 + mn$.

2. For the "torus-graph" $C_{2n} \times C_{2m}$ we have shown that $\beta(C_{2n} \times C_{2m}) = mn + i$, where $i = 2$, 3, or 4. However, we have not been successful in specifying the conditions which distinguish these three values.

We believe that a method for determining the bichromaticity of bigraphs may often be successfully developed in two stages. First, inequalities arising from the lemma may be used to give bounds on the parameters r and s. Second, an *ad hoc* argument depending on the class of bigraphs in question will then yield an exact or "nearly" exact formula for the bichromaticity.

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University of Michigan,
Ann Arbor, Michigan,
U.S.A.