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Lattice bandwidth of random graphs

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Abstract

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The bandwidth of a random graph has been well studied. A natural generalization of bandwidth involves replacing the path as host graph by a multi-dimensional lattice. In this paper we investigate the corresponding behavior for random graphs.

1. Introduction and theorem

Let G and H be graphs with the same number of vertices. Given a bijection ϕ from $V(G)$ to $V(H)$ let $|\phi|$ denote the maximum, over all edges xy of G, of the distance in H between $\phi(x)$ and $\phi(y)$. Now let $B(G,H)$ be the minimum value of $|\phi|$ over all such ϕ.

The investigation of $B(G,H)$ for certain classes of "host graphs" H, apart from its intrinsic interest as a graph theory problem, is motivated by issues in VLSI and parallel computation. In VLSI applications the parameter $B(G,H)$ gives a lower bound for the length of wires when an electronic circuit (modelled by G) is embedded on a chip (modelled by H) [14,15,8]. In parallel computation we may have an algorithm A designed to run on a network of processors G, and instead we wish to run A on a different network H. Here $B(G,H)$ is a lower bound on the communication delay per unit step when we simulate G by H [9,12].

When H is a path the parameter $B(G,H)$ is known as the *bandwidth* $B(G)$ of G. The motivation for studying $B(G)$ arose first in numerical analysis as follows. Given a symmetric $n\times n$ matrix M with 0's on the diagonal, one may wish to perform a symmetric permutation of the rows and columns of M with a view to bringing the nonzero entries of the resulting matrix into as narrow a band as possible about the diagonal. The reason for doing this is that there are algorithms for certain matrix operations, such as Gaussian elimination and matrix inversion, which work fastest when this band is narrow. Now let $G(M)$ be the graph on n vertices $\{1,2,\dots,n\}$ with i and j joined by an edge if and only if the (ij)th entry of M is nonzero. Then $B(G(M))$ is the width of the smallest possible band achievable.

It is well known that the problem of computing $B(G)$ is NP-complete, even when G is a tree of maximum degree three [6]. This has prompted work on the probabilistic analysis of this problem. Turner [16] explains the success of some well-known heuristics from a probabilistic point of view. Kuang and McDiarmid [7] show that for a random graph G with fixed edge probability p we have

$$B(G)=n-(2+2^{1/2}+o(1))\frac{\log(n)}{\log(1/(1-p))} \tag{1.1}$$

with probability approaching 1 as $n\to\infty$. A similar though less precise result appears in [5]. Sparse random graphs are considered in [17].

Here we consider the case when H is a multi-dimensional lattice or grid. Let $[n]^k$ denote the graph with vertex set $\{(x_1,x_2,\dots,x_k)\colon 0\le x_i\le n,\ x_i \text{ integer}\}$ and an edge between vertices x and y of $[n]^k$ if and only if $\sum_{i=1}^{k}|x_i-y_i|=1$. Note that this graph $[n]^k$ has $(n+1)^k$ vertices and diameter kn.

The NP-completeness of determining $B(G,[n]^k)$ when $k=2$ is shown independently in [11,2,1], and the proofs extend readily to arbitrary dimension k. Bounds for $B(G,[n]^k)$ may be derived from [14,15], and further work by these and other authors. We wish to investigate the "usual" behavior of $B(G,[n]^k)$.

We take as our probability model the set $\mathcal{G}(n,p)$ of all labelled graphs on n points, with labels from $\{1,2,\dots,n\}$, having edge probability p where p is fixed. We set $q=1-p$. Under this model the probability that two points i and j are joined by an edge in G is p, and these events are independent. Let A be any property of graphs, and let A_n be its restriction to graphs on n points. We say that A happens *almost surely* if $\mathrm{Prob}(A_n)\to 1$ as $n\to\infty$. For applications of this model to many different problems in graph theory the reader is referred to [3,13].

Let $n=n(t)$ and $k=k(t)$ be functions from the positive integers $\{1,2,\dots\}$ to the positive integers. Define $N=N(t)=(n+1)^k$. We are interested in the random variables $B_t=B(G,[n]^k)$, $t=1,2,\dots$, where $G\in\mathcal{G}(N,p)$. We shall always assume that $n+k\to\infty$ (or equivalently $N\to\infty$) as $t\to\infty$. At one extreme we could have $k=1$, $n\to\infty$ (the path) and at the other $n=1$, $k\to\infty$ (the k-dimensional cube).

Define the functions w and b by

$$w=w(t)=2+k/\log\log(n+2)\quad(\text{so } w>2),$$

and

$$b=b(t)=1+k(\log n)^{1/k}/w\log(w)$$

(all logarithms are natural).

The ratio w measures how "cube-like" the lattice graph $[n]^k$ is. The smaller w is the more $[n]^k$ is like the path, and the larger w is the more it is like the cube. We discuss the function b after the following theorem, which is our main result.

Theorem. *Let $G \in \mathcal{G}(N,p)$, where $N = |[n]^k| = (n+1)^k$. Then there are positive constants c_1 and c_2 such that as $t \to \infty$ we have*

$$\operatorname{Prob}(nk - c_1 b < B(G,[n]^k) < nk - c_2 b) \to 1.$$

In order to gain some feeling for the result let us briefly consider the behavior of b depending on w. To do this we recall some notation on growth rates. Let $f = f(t)$ and $g = g(t)$ be functions defined for positive integers t and taking positive values. We write $f = \mathrm{O}(g)$ or $g = \Omega(f)$ if f/g is bounded above by a constant, we write $f = \mathrm{o}(g)$ if $f/g \to 0$ as $t \to \infty$; and we write $f = \Theta(g)$ if we have both $f = \mathrm{O}(g)$ and $g = \mathrm{O}(f)$. Our theorem could thus be stated as

$$B_t = nk - \Theta(b) \quad \text{almost surely.}$$

Observe the following asymptotic properties of the function b (as $t \to \infty$).

(i) If $w = \mathrm{O}(1)$ as $t \to \infty$, then $n \to \infty$, $b \to \infty$, and $b = \Theta(k(\log n)^{1/k})$. In particular if $w = \Theta(1)$, then $b = \Theta(k) = \Theta(\log\log(n))$.

(ii) If w_0 is a constant >2, and $w \geq w_0$ but $w = \exp(\mathrm{O}(\log\log(n+2)))$, then $n \to \infty$, $k \to \infty$, and $b = \Theta(k/w \log w) = \Theta(\log\log(n)/\log w)$.

(iii) If $w = \exp(\Omega(\log\log(n+2)))$, then $k \to \infty$ and $b = \Theta(1)$.

2. Proof of Theorem

We define an *extreme point* of the graph $[n]^k$ as a vertex v with each coordinate 0 or n. The point *opposite* to v is the extreme point $z = (n, n, \ldots, n) - v$, that is, the point whose ith coordinate z_i is 0 (respectively n) when v_i is n (respectively 0).

Note that if $r \leq n$, then the number of vertices of $[n]^k$ at distance at most r from the origin (and thus from any given extreme point) equals

$$\sum_{i=0}^{r} \binom{i+k-1}{k-1} = \binom{r+k}{k}.$$

The quantity $\binom{r+k}{k}$ will turn up frequently below, and when it does r will always denote a nonnegative integer.

For any graph G let $f(G)$ be the maximum integer m such that G contains disjoint sets S and T such that $|S| = |T| = m$ and there is no edge between S and T.

Lemma 2.1. *Let the graph G have $(n+1)^k$ vertices.*

(a) *If $f(G) < \binom{r+k}{k}$, then $B(G,[n]^k) \geq nk - 2r$.*

(b) *If $r < \frac{1}{2}n$ and G has 2^{k-1} pairwise disjoint independent sets of size $2\binom{r+k}{k}$, then $B(G,[n]^k) < nk - r$.*

Proof. (a) We may assume that $r<nk/2$. Consider any pair u, v of opposite extreme points of $[n]^k$. Let U and V be the sets of vertices at distance at most r from u and v respectively. Then $U\cap V=\emptyset$ and $|U|=|V|=\binom{r+k}{k}$. Now let ϕ be any bijection of $V(G)$ onto $V([n]^k)$. Then by the definition of $f(G)$ we know that there must exist an edge between $\phi^{-1}(U)$ and $\phi^{-1}(V)$, and hence $|\phi|\geq nk-2r$.

(b) List the given independent sets in G as $S_1, S_2, \ldots$, and list the 2^{k-1} pairs of opposite extreme points of $[n]^k$ as $\{u^{(1)}, v^{(1)}\}$, $\{u^{(2)}, v^{(2)}\}, \ldots$. For each $i=1,2,\ldots$ let T_i be the set of vertices of $[n]^k$ at distance at most r from $u^{(i)}$ or from $v^{(i)}$. Note that the sets T_i are pairwise disjoint, and each is of size $2\binom{r+k}{k}$. Now define a bijection ϕ of G into $[n]^k$ by mapping each subset S_i of G onto the subset T_i of $[n]^k$ in any way, and then completing ϕ arbitrarily.

We shall now see that $|\phi|<nk-r$. Let s and t be any two vertices of $[n]^k$ at distance $d(s,t)\geq nk-r$. Thus $\sum_{i=1}^k (n-|s_i-t_i|)\leq r$, so there must be opposite extreme points $u^{(i)}$, $v^{(i)}$ in $[n]^k$ such that $d(s,u^{(i)})+d(t,v^{(i)})\leq r$. Hence $\{s,t\}\subset T_i$ so $\{\phi^{-1}(s),\phi^{-1}(t)\}\subset S_i$, and thus $\phi^{-1}(s)$ and $\phi^{-1}(t)$ are not adjacent in G. Since s and t were arbitrary, this shows that $|\phi|<nk-r$ as required. □

The above lemma applied to random graphs yields the lemma on which our proof of the theorem rests. Recall that $q=1-p$.

Lemma 2.2. *Let $r=r(t)$ be a nonnegative integer.*

(a) *If $\binom{r+k}{k}\geq 2\log N/\log(1/q)$, then $B_t\geq nk-2r$ almost surely.*

(b) *If $r<\frac{1}{2}n$, and $\binom{r+k}{k}\leq\frac{2}{3}\log N/\log(1/q)$, then $B_t<nk-r$ almost surely.*

Proof. (a) Let $s=s(t)=\lceil 2\log N/\log(1/q)\rceil$. Then for $G\in\mathscr{G}(N,p)$ we have

$$\begin{aligned}\text{Prob}(f(G)\geq s) &\leq \frac{1}{2}\binom{N}{s}\binom{N-s}{s}(1-p)^{s^2}\\ &< \left(\left(\frac{Ne}{s}\right)^2(1-p)^s\right)^s\\ &\to 0 \quad \text{as } t\to\infty.\end{aligned}$$

The proof is then completed by applying Lemma 2.1(a).

(b) Note first that if $n=1$, then $r=0$; and by Lemma 2.1(b), it suffices now to recall that the complement of our random graph almost surely contains a perfect matching (see for example [3]).

We may thus assume that $n\geq 2$ (for all t). Let $0<\varepsilon<1$ and let $s=\lceil(1-\varepsilon)\log N/\log(1/q)\rceil$. Then $2^{k-1}s=o(N)$, and it follows from recent results of Bollobás [4] that almost surely there are 2^{k-1} pairwise disjoint independent sets of size $2s$. Lemma 2.1(b) now completes the proof. □

We remark that if in part (b) we are willing to accept a factor of $\frac{1}{3}$ instead of $\frac{2}{3}$ we may simply consider the greedy coloring algorithm (see [10]).

We are now ready to prove the theorem. We will consider four size ranges for w.

(1) *Range w small* ($2<w\leq w_0$, where w_0 is a suitably small constant).

If w is small, then b is about $k(\log n)^{1/k}$ and $(\log n)^{1/k}$ is large. Thus we may choose a sufficiently small absolute constant $w_0>2$ (not depending on the functions $n(t)$ or $k(t)$, or on t) such that the following holds for $w\leq w_0$. Let $\alpha=\min\{1, 1/(3\log(1/q))\}$, and set

$$r=\left\lfloor \frac{\alpha}{3e}b \right\rfloor \leq \frac{\alpha}{2e}k(\log n)^{1/k}$$

Then

$$\binom{r+k}{k}\leq\left(e\left(1+\frac{r}{k}\right)\right)^k\leq\left(e\,\frac{\alpha(\log n)^{1/k}}{e}\right)^k=\alpha^k(\log n)\leq\frac{1}{3}\frac{\log N}{\log(1/q)}.$$

Hence by Lemma 2.2(b) we have $B_t<nk-r$ almost surely.

Next let β be a suitably large constant, and set $r=\lceil 2\beta b\rceil\geq\beta k(\log n)^{1/k}$. Then

$$\binom{r+k}{k}\geq\left(\frac{r}{k}\right)^k\geq\beta^k(\log n)>3\frac{\log N}{\log(1/q)}.$$

Hence by Lemma 2.2(a) we have $B_t\geq nk-2r$ almost surely.

(2) *Range w very big* ($w\geq(\log(n+1))^{1/2}$).

Suppose $w\geq(\log(n+1))^{1/2}$. Now $b=\Theta(1)$ by observation (iii) in the first section. But by Lemma 2.2(b) with $r=0$ we have $B_t\leq nk-1$ almost surely.

For the lower bound note that with $r=4$ we have

$$\binom{r+k}{k}\geq\frac{k^4}{24}\geq\Omega(k^2\log(n+1))=\Omega(k\log N).$$

Now use Lemma 2.2(a).

(3) *Range w big* ($w_1\leq w\leq(\log(n+1))^{1/2}$, with w_1 a suitably large constant).

Set $r=c\log\log(n+2)/\log w$, where $\frac{1}{2}\leq c\leq 2$. Thus $b=\Theta(r)$ by observation (ii). Now $k/r=(1/c)(w-2)\log w$, and so when w is large we have $k\gg r$. It follows that for a suitably large constant w_1, for all $w\geq w_1$, and all $\frac{1}{2}\leq c\leq 2$ we have

$$\binom{r+k}{k}=(\log n)^c(1+\alpha)\quad\text{for some }\alpha,\ 0\leq\alpha\leq 1. \tag{$*$}$$

Now set $c=\frac{1}{2}$ and use Lemma 2.2(b) to get the required almost sure upper bound on B_t. For the lower bound observe that

$$\log N=k\log(n+1)\leq(\log(n+1))^{3/2}\log\log(n+2)=o((\log n)^2).$$

Now setting $c=2$ in ($*$) above we get the required lower bound by Lemma 2.2(a).

(4) *Range w intermediate* ($w_0\leq w\leq w_1$, for the constants w_0 and w_1).

Now by observation (i) we have $k=\Theta(\log\log(n))$ and $b=\Theta(k)=\Theta(\log\log(n))$.

Setting $r \sim ck$ for a positive constant c, we have

$$\log\binom{r+k}{k} = ck \log\left(1+\frac{1}{c}\right) + k\log(1+c) + O(\log(k)).$$

To obtain the upper bound for B_t, note that if c is sufficiently small, then $\log\binom{r+k}{k} \leq \frac{1}{2}\log\log(n)$ for n large enough, and then use Lemma 2.2(b). To obtain the lower bound for B_t, note that if c is sufficiently large, then $\log\binom{r+k}{k} \geq 2\log\log(n)$ for n large enough, and then use Lemma 2.2(a).

The four ranges for w fit together to complete the proof of the theorem. □

3. Concluding remarks

We note that essentially the same proof shows the following. Let $r^* = r^*(t)$ be the smallest integer $r \geq 1$ satisfying $\binom{r+k}{k} \geq \log N$. Then

(a) $B_t = nk - \Theta(r^*)$ almost surely, and
(b) $r^* = \Theta(b)$.

We know that $B_t = nk - \Theta(b)$ almost surely, but can we be more precise?

The case $k=1$, $n \to \infty$ concerns the bandwidth of a random graph, and here the behavior of B_t is known rather exactly; see (1.1). Other cases of particular interest are $k=2$, $n \to \infty$ (the square lattice) and $n=1$, $k \to \infty$ (the k-dimensional hypercube).

Square lattice: $k=2$, $n \to \infty$. From our theorem we have $B_t = 2n - \Theta((\log(n))^{1/2})$ in probability. Will the methods in [7] for the bandwidth yield an exact result like (1.1)?

Hypercube: $n=1$, $k \to \infty$. From our theorem we know that for some constant c, $k - c \leq B_t \leq k-1$ almost surely. It is easy to improve on this. Indeed, we have

(3.1) $B_t = k-1$ or $k-2$ almost surely for any fixed p, and
(3.2) $B_t = k-1$ almost surely for $p \geq \frac{3}{4}$.

To prove (3.1) we must show that $\text{Prob}(B_t < k-2) \to 0$ as $t \to \infty$. Using Markov's inequality this can be done by checking that

$$(2^k)!\, q^{2^{k-1}}\left(1 + k + \binom{k}{2}\right) \leq 2^{k2^k} q^{k^2 2^{k-2}} \to 0 \quad \text{as } k \to \infty.$$

We may prove (3.2) in a similar way by considering $(2^k)!\, q^{2^{k-1}(1+k)}$.

The interesting question is what happens when $p = \frac{1}{2}$. Can we improve on (3.1)?

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