Chromatic Numbers of Hypergraphs and Coverings of Graphs

Zevi Miller
MIAMI UNIVERSITY
OXFORD, OHIO
Heinrich Müller
UNIVERSITÄT STUTTGART
STUTTGART, WEST GERMANY

ABSTRACT

Burr recently proved [3] that for positive integers $m_1, m_2, \dots, m_k$ and any graph G we have $\chi(G) \le \prod_{i=1}^{k} m_i$ if and only if G can be expressed as the edge disjoint union of subgraphs F_i satisfying $\chi(F_i) \le m_i$. This theorem is generalized to hypergraphs. By suitable interpretations the generalization is then used to deduce propositions on coverings of graphs.

1. THE THEOREM

In a recent paper [3], Burr proves a result which relates the chromatic number $\chi(G)$ of a graph G to the chromatic numbers $\chi(F_i)$ of the subgraphs F_i appearing in an edge-disjoint factorization of G. Our purpose in this paper is twofold. First we generalize Burr's theorem to hypergraphs, and second we apply this generalization to obtain results on coverings of graphs by subgraphs possessing specified properties. For graph or hypergraph terminology not given here we refer to [1], [2], or [5].

We begin with Burr's result.

Theorem 1 [3]. Let G be a graph, and $m_1, m_2, \dots, m_k$ positive integers. Then G is the edge disjoint union, $G = \bigcup_{i=1}^{k} F_i$, of subgraphs F_i with $\chi(F_i) \le m_i$, $1 \le i \le k$, if and only if $\chi(G) \le \prod_{i=1}^{k} m_i$.

Journal of Graph Theory, Vol. 5 (1981) 299–305
CCC0369-9024/81/030299-07$01.00

To state our result we recall some terminology of hypergraphs. A finite *hypergraph* $H = (\chi, \mathcal{E})$ consists of a finite, nonempty set X of nodes and a finite set $\mathcal{E}$ of edgės, where $\mathcal{E}$ is a subset of power set of X. The *chromatic number* $\chi(H)$ is defined to be the minimum number of colors needed to color the nodes of H so that no edge with more than one element has all its nodes of the same color. A hypergraph $H' = (X', \mathcal{E}')$ with $X' = X$ and $\mathcal{E}' \subseteq \mathcal{E}$ is called a *partial hypergraph* of H.

We may now state our result.

Theorem 2. Let H be a hypergraph, and $m_1, m_2, \ldots, m_k$ positive integers. Then H is the edge disjoint union, $H = \bigcup_{i=1}^{k} H_i$ of hypergraphs $H_i$with $\chi(H_i) \leq m_i$, $1 \leq i \leq k$ if and only if $\chi(H) \leq \prod_{i=1}^{k} m_i$.

Proof. Let $H_1, \ldots, H_k$ be partial hypergraphs covering the edges of H, $\chi(H_i) \leq m_i$. Set $N = \prod_{i=1}^{k} m_i$. We will define an N-coloring of $H = (X, \mathcal{E})$ as follows. Consider the set of k-tuples $T = \{(r_1, \ldots, r_k) \mid 1 \leq r_i \leq m_i\}$, and assign to each node $x \in X$ a k-tuple $(r_1(x), \ldots, r_k(x))$ in T by letting $r_i(x)$ be the color (viewed as a number) of x in an m_i-coloring of H_i. To show that this is an N-coloring of H, we show that no edge $E \in \mathcal{E}$ with $|E| \geq 2$ has all its nodes assigned the same k-tuple. Let $E \in \mathcal{E}$ be arbitrary, and observe that E is an edge of H_i for some i, $1 \leq i \leq k$. Hence there are nodes $x_1, x_2 \in E$ having distinct ith coordinates in their k-tuples. In particular, the k-tuples of x_1 and x_2 are distinct, and hence we have an N-coloring of H as desired.

Given $H = (X, \mathcal{E})$ with $\chi(H) \leq \prod_{i=1}^{k} m_i$, consider again the set T of k-tuples defined above. Since $|T| = N = \prod_{i=1}^{k} m_i$, any N-coloring of it may be viewed as an assignment of the k-tuples in T to the nodes of H so that no edge $E \in \mathcal{E}$, $|E| \geq 2$, has all its nodes assigned to the same k-tuple. We construct partial hypergraphs H_i, $1 \leq i \leq k$, with $\chi(H_i) \leq m_i$ as follows. Let $E \in \mathcal{E}$, $|E| \geq 2$. Then there exist a pair of nodes $x_1, x_2 \in E$ such that the k-tuples of x_1 and x_2 differ. Let $c(x_1, x_2)$ be the first coordinate in which these k-tuples differ, and give E the color $c(E)$, where $c(E) = \min\{c(x_1, x_2) \mid x_1, x_2 \in E\}$. For $E \in \mathcal{E}$, $|E| = 1$, set $c(E) = 1$. Consider the set of integers $S = \{c(E) \mid E \in \mathcal{E}\} \subseteq \{1, \ldots, k\}$. Now for $s \in S$ the partial hypergraph $H_s = (X_s, \mathcal{E}_s)$ of H, with $E \in \mathcal{E}_s$ if and only if $c(E) = s$, satisfies $\chi(H_s) \leq m_s$. For $s \notin S$, $1 \leq s \leq k$, take $H_s = (X_s, \mathcal{E}_s)$ with $\mathcal{E}_s = \phi$. Then the partial hypergraphs $H_1, \ldots, H_k$ form an edge cover of H with the required property. ■

Of course Burr's theorem follows as a corollary. We also have the following consequence. For a hypergraph H, define $\beta(H)$ to be the minimum number of 2-colorable partial hypergraphs of H which cover the edges of H.

Corollary 2.1. For any hypergraph H we have $\beta(H) = \{\log_2 \chi(H)\}$.

The special case of this corollary in which H is a graph was proved in [6] and independently in [7].

2. CONSEQUENCES FOR GRAPHS

Given a hypergraph $H = (X, \mathrm{E})$, an *edge cut* of H consists in deleting all edges in H which contain nodes in both of the two sets in a given partition of $X(S_0, S_1 \neq \phi, S_0 \cup S_1 = X, S_0 \cap S_1 = \phi)$. A *node cut* of H consists in first replacing each node x of H which is contained in an edge, by two nodes x^0, x^1, then replacing each edge $E = \{x_1, \ldots x_m\}$ of H by either $E^0 = \{x_1^0, \ldots, x_m^0\}$ or $E^1 = \{x_1^1, \ldots, x_m^1\}$, and finally deleting all nodes of the form x^0 or x^1 which are not contained in an edge. Here we only consider node cuts not having all the new edges with the same upper index.

The notions of edge cut and node cut are illustrated in Figures 1 and 2.

Theorem 3. Let G be a graph, P a graph property satisfying

(1) P is hereditary on induced subgraphs;
(2) if each connected component has property P, then so does the whole graph;
(3) a single node has property P.

Let $\chi_p^n(G)$ be the minimum number of induced subgraphs having property P which cover the nodes of G, $c_p^e(G)$ the minimum number of edge cuts which are necessary so that the resulting graph has property P. Then

$$c_p^e(G) = \{\log_2 \chi_p^n(G)\}.$$

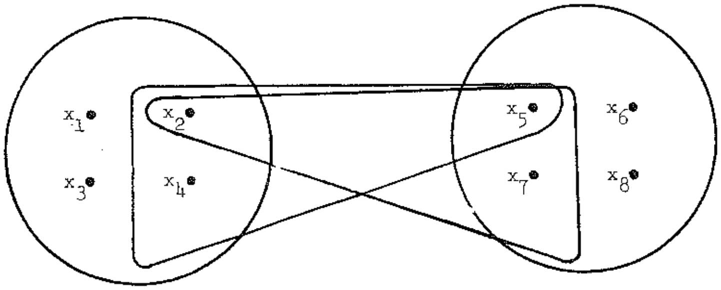
Proof. Given a graph $G = (X, \mathrm{E})$ and a property P, let $H' = (X', \mathrm{E}')$ be the hypergraph with $X' = X$ and $\mathrm{E}' = \{E' \mid E' \subseteq X'$, and the subgraph of G induced by E' is minimal in not having property $P\}$. By (3) each minimal graph not having property P has at least two nodes, and by (1), an induced subgraph of G has property P if and only if it has no minimal induced subgraph not having property P. So we have $\chi(H') = \chi_p^n(G)$. Let $c(H')$ be the minimum number of edge cuts necessary to delete all edges of H'. Then we get $c(H') = \beta(H')$, for the edges deleted by an edge cut in H' induce a 2-colorable partial hypergraph of H', and to each 2-colorable partial hypergraph there is an edge cut in H' deleting at least its edges. Hence by Corollary 2.1 we have $c(H') = \{\log_2 \chi_p^n(G)\}$. Now it suffices to show $c_p^e(G) = c(H')$.

First, by (2), we note that a minimal graph G_0 not having property P is connected. So each edge cut in G_0 produces a graph having property P.

a. Given hypergraph and point partition

$S_0 = \{x_1, x_2, x_3, x_4\}$, $S_1 = \{x_5, x_6, x_7, x_8\}$

$E = \{\{x_1,x_2,x_3,x_4\}, \{x_5,x_6,x_7,x_8\}, \{x_2,x_4,x_5\}, \{x_2,x_5,x_7\}\}$



b. Resulting hypergraph after edge cut

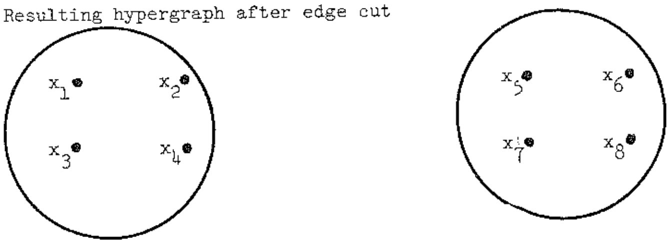


FIGURE 1. An edge cut applied to a hypergraph

Hence, given a sequence of $c(H')$ edge-cuts in H', so that all edges are deleted, the sequence of edge cuts in G having the same partitions of the node set produces a graph which has no subgraph not having property P. Hence $c(H') \geq c_p^e(G)$. Given a sequence of $c_p^e(G)$ edge cuts in G, so that the resulting graph has property P, the sequence of edge cuts in H' having the same partitions of the node set deletes all edges in H', hence $c_p^e(G) \geq c(H')$. So we have $c_p^e(G) = c(H')$ as desired.∎

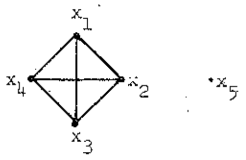
For coverings of the edge set we have

Theorem 4. Let G be a graph, P' a graph property satisfying

(1) P' is hereditary on subgraphs,
(2) if each connected component has property P', then so does the whole graph,
(3) a single edge has property P'.

Let $\chi_{p'}^e(G)$ the minimum number of subgraphs having property P' which are necessary to cover the edges of G, $c_{p'}^n(G)$ the minimum number of node

a. Given hypergraph

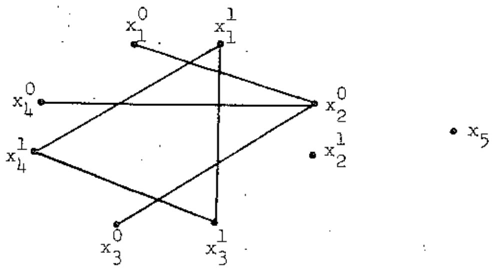


b. Duplicating of nodes and replacing the edges

$\{x_1, x_2\} \rightarrow \{x_1^0, x_2^0\}$ $\{x_1, x_4\} \rightarrow \{x_1^1, x_4^1\}$

$\{x_2, x_3\} \rightarrow \{x_2^0, x_3^0\}$ $\{x_1, x_3\} \rightarrow \{x_1^1, x_3^1\}$

$\{x_3, x_4\} \rightarrow \{x_3^1, x_4^1\}$ $\{x_2, x_4\} \rightarrow \{x_2^0, x_4^0\}$



c. Deleting nodes with upper index to get resulting hypergraph

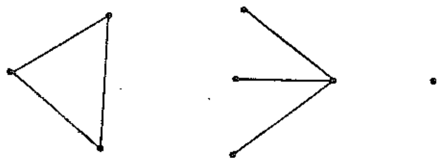


FIGURE 2. A node cut applied to a hypergraph

cuts which are necessary so that the resulting graph has property P'. Then $\mathcal{C}_{P'}^n(G) = \{\log_2 \chi_{P'}^e(G)\}$.

Proof. Given a graph $G = (X, \mathsf{E})$ and property P', let $H' = (X', \mathsf{E}')$ be the hypergraph with $X' = \mathsf{E}$ and $\mathsf{E}' = \{E' \mid E' \subseteq X'$, and the subgraph of G induced by the edges in E' is minimal in not having property $P'\}$. As in the proof of Theorem 3 we have $c(H') = \beta(H') = \{\log_2 \chi_{P'}^e(G)\}$. Now we show $\mathcal{C}_{P'}^n(G) = c(H')$.

By (2) we note that a minimal graph G_0 not having property P' is connected, and by (3) that G_0 has at least two edges. So each node cut in G_0 produces a graph whose components are subgraphs of G_0 not equal to G_0. Hence this graph has property P'. To each edge cut in H' with node set partition S_0, S_1, there corresponds a node cut of G defined by replacing $E \in \mathsf{E}$ by E^0, if $E \in S_0$, and by E^1, if $E \in S_1$. Thus, to a given sequence of $c(H')$ edge cuts in H' deleting all edges in H', the sequence of the corresponding node cuts in G produces a graph which has no subgraph not having property P'. Hence $c(H') \geq c^n_{p'}(G)$. To each node cut in G there corresponds an edge cut in H' defined by the partition S_0, S_1, with $S_0 = \{E \mid E \in \mathsf{E}$, and E^0 is an edge in the resulting graph of the node cut$\}$, $S_1 = X' - S_0$. Therefore, given a sequence of $c^n_{p'}(G)$ node cuts in G, so that the resulting graph has property P', the sequence of the corresponding edge cuts in H' deletes all edges in H'. We therefore get $c^n_{p'}(G) \geq c(H')$, and hence $c^n_{p'}(G) = c(H')$ as desired. ∎

Many well known graph properties are of these forms. For example, if $P =$ "planar," then $\chi^n_{\text{planar}}(G)$ is the minimum number of induced planar subgraphs which are necessary to cover all the nodes of a graph G. Then by Theorem 3 we get the following proposition.

Corollary 4.1. Let G be a graph. Then the minimum number of edge cuts which are necessary so that the resulting graph is planar, is $\{\log_2 \chi^n_{\text{planar}}(G)\}$.

If $P' =$ "bipartite," we get the following.

Corollary 4.2. Let G be a graph. Then the minimum number of node cuts which are necessary so that the resulting graph is bipartite, is $\{\log_2 \log_2 \chi(G)\}$.

Proof. By Theorem 4 the minimum number of node-cuts is $\{\log_2 \chi^e_{\text{bipartite}}(G)\}$. By Corollary 2.1 we have $\chi^e_{\text{bipartite}}(G) = \{\log_2 \chi(G)\}$. This implies the corollary by observing that $\{\log_2 \{\log_2 \chi(G)\}\} = \{\log_2 \log_2 \chi(G)\}$. ∎

So for example the four-color-theorem is equivalent to the proposition that for any planar graph there is a node cut so that the resulting graph is bipartite.

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