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THE BICHROMATICITY OF A TREE

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Abstract

The bichromaticity $\beta(B)$ of a bipartite graph B is defined as the maximum order of a complete bipartite graph onto which B is homomorphic. It is specified that no two points of different colors can be sent to the same point under a homomorphism. Exact formulas are obtained for the bichromaticity of a tree and for that of even cycles.

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1. Bichromaticity

Our object is to study for bipartite graphs B the invariant corresponding to the achromatic number $\psi(G)$ of an arbitrary graph G which was introduced in [2]. This was defined as the maximum order p of a complete graph K_p onto which G is homomorphic. Analogously, the bichromaticity $\beta(B)$ is the maximum order $p = r+s$ of a complete bigraph $K_{r,s}$ onto which B is homomorphic, no two points in B of different colors being sent to the same point.

An elementary homomorphism will mean as usual the identification of two points in the same color class. A homomorphism of a bigraph will then be a sequence of elementary homomorphisms. A homomorphism $h: B \to K_{r,s}$ will be called bicomplete.

Let B denote a connected bigraph with color classes C and D so that its point set is $V = C \cup D$. Without loss of generality, we stipulate that $|C| \geq |D|$. We then call C the majority of B and D the minority, and we write $\mu = \mu(B) = |C|$. Furthermore, whenever we say that a bipartite graph is complete, we will mean that it is a complete bigraph. For other notation and terminology, we follow the book [1].

so $s \leq (p-1)/r$. But by (2) ,

$$r + s \geq \mu + 1 ,$$

so that $(p-1)/r + r \geq \mu + 1$. As we obviously have $\mu \geq p/2$, it follows that $p(2-r)/2r \geq 1 - r + 1/r$. We now show that $r \geq 4$ leads to a contradiction as follows. If $r \geq 4$, the above inequality becomes after routine manipulation

$$\frac{p}{r} \leq 2 - \frac{2}{2-r} + \frac{2}{r(2-r)} .$$

But we also have

$$(0) \qquad s \leq \frac{p-1}{r} \leq 2 - \frac{2}{2-r} + \frac{2}{r(2-r)} - \frac{1}{r} .$$

For $r \geq 4$, this gives $s \leq 2 + 2/(r-2) \leq 3$, contradicting the convention $r \leq s$.

We now examine the case $r = 3$. Substituting into (0), we get $s \leq 3$. By our convention, we then have $r = s = 3$. This leaves $r \leq 2$ as required.

Remark: It is possible to characterize those trees for which $r = s = 3$ in the preceeding lemma as follows.

For such a tree T , there is a homomorphism $h\colon T \to K_{3,3}$ and $\beta(T) = 6$. The tree T must have at least 10 points since otherwise $q(T) \leq 8$. But Lemma 1a gives $q(T) \geq r \cdot s = 9$, a contradiction. Now since $p(T) \geq 10$, we have $\mu(T) \geq 5$. Therefore, $\mu(T) \geq 6$ or $\mu(T) = 5$. If $\mu(T) \geq 6$, we derive a contradiction

Summarizing, we have shown that the trees T for which $r = s = 3$ in the lemma must satisfy the following properties:

$$p = 10, \quad \mu = 5, \quad \Delta \leq 3, \quad \text{and}$$

T has at least two points of degree 3.

A check of Appendix 3 in [1] listing all the trees with ten points shows that only sixteen satisfy these properties. Of those sixteen, we have verified that only the seven shown in Figure 1 have the $r = s = 3$ property.

We are now ready to determine the bichromaticity of an arbitrary tree T.

Theorem 1. For any tree T, $\beta(T) = \mu + 1$.

Proof: By Lemma 1b, if $r + s = \beta(T)$ is the maximum order of a complete bigraph $K_{r,s}$ onto which T is homomorphic, then $r = 3$ and $s = 3$, or $r \leq 2$. We begin by considering the first possibility.

As shown in the remark following Lemma 1b, $r = 3$ and $s = 3$ implies $\mu(T) = 5$. But then $\mu(T) + 1 = 5 + 1 = 6 = r + s = \beta(T)$, so the theorem is proved in this case. We now pass to a consideration of the case $r \leq 2$.

We may decompose the point set $V(T) = C \cup D$ uniquely into two color classes C and D. As above, we again take $|C| \geq |D|$.

We note first that tight bounds can be given for $\beta(T)$ as follows. Let $h: T \to K_{r,s}$ be a bicomplete homomorphism for which $r + s = \beta(T)$. By Lemma 1a and 1b and the fact that $s \leq \mu$, it follows that $\mu + 1 \leq \beta(T) \leq \mu + 2$. It only remains to show that $\beta(T) < \mu + 2$.

Suppose to the contrary that $\beta(T) = \mu + 2$. Then since $s \leq \mu$ and $r \leq 2$, we must have $s = \mu$ and $r = 2$. Thus the homomorphism h sends the points of D onto two points v and w, but h leaves the points of C fixed.

Our plan is to show that in order to derive a contradiction to $\beta(T) = \mu + 2$, it will suffice to prove that C must contain at least one endpoint e of T. For if this holds, let x_0 be the point adjacent to e and partition set D so that $X_1 = \{x \in D/h(x) = v\}$ and $X_2 = \{x \in D/h(x) = w\}$. Then since $D = X_1 \cup X_2$ and $x_0 \in D$, either $x_0 \in X_1$ or $x_0 \in X_2$. Without loss of generality we assume $x_0 \in X_1$. Then the endpoint e is not adjacent to any point of X_2, so e is not adjacent to w in $h(T)$, contradicting the completeness of $h(T)$.

To prove that C does indeed contain an endpoint of T, suppose to the contrary that all the endpoints of T are in D. As usual, define $T^{(k)}$ inductively as the subtree obtained from $T^{(k-1)}$ by deleting all its endpoints, with $T^{(0)} = T$. We find it convenient to take $T^{(k)}$ as empty when $T^{(k-1)}$ is either K_2 or K_1. Also write $U^{(k)}$ for the set of endpoints of $T^{(k)}$ and let N be the smallest integer such that $T^{(N)} = \phi$.

The theorem is easily specialized to handle paths.

Corollary 1a. The bichromaticity of the path P_n of order n is given by

$$\beta(P_n) = \begin{cases} 2 + [n/2] & n \text{ odd} \\ 1 + n/2 & n \text{ even} \end{cases} \tag{3}$$

Theorem 2. The bichromaticity of the even cycle C_{2n} is given by

$$\beta(C_{2n}) = \begin{cases} 2 + n & \text{if } n \text{ is even} \\ 1 + n & \text{if } n \text{ is odd} \end{cases} \quad . \tag{4}$$

Proof: As in the case of trees, we first give tight bounds for $\beta(C_{2n})$. Since $\mu(C_{2n}) = n$, Lemma 1a gives $\beta(C_{2n}) \geq n + 1$. On the other hand, Lemma 2a and $s \leq \mu$ give $\beta(C_{2n}) \leq n + 2$. Thus

$$n + 1 \leq \beta(C_{2n}) \leq n + 2 \quad . \tag{5}$$

Consider first the case that n is even. Going around the cycle in either of the two possible directions, number the points traversed $v_1, v_2, \ldots, v_{2n}$ successively. We partition $V(C_{2n}) = C \cup D$ so that $C = \{v_k \mid k \text{ odd}\}$ and $D = \{v_k \mid k \text{ even}\}$. We now describe index $j + 2$ will be read modulo $2n$. Thus in $h''(C_{2n})$, v_{j+1} is not adjacent to w, contradicting the completeness of $h''(C_{2n})$ and proving (4) for n odd.

the parameters r and s in a bicomplete homomorphism $h\colon B \to K_{r,s}$ for which $\beta(B) = r + s$ are uniquely determined. For instance $\beta(C_6) = 4$ according to Theorem 2. But this number can arise via homomorphisms

$$h_1\colon C_6 \to K_{1,3} \text{ or } h_2\colon C_6 \to K_{2,2} .$$

4. Lemma 1a states that $\beta \geq 1 + \mu$ for any bigraph B. Then Theorems 1 and 2 assure that equality holds for trees and for even cycles C_{2n} with n odd. It is natural to ask for a characterization of bigraphs for which $\beta = 1 + \mu$.

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