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Cycles of given lengths in hypergraphs

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ABSTRACT

In this paper, we develop a method for studying cycle lengths in hypergraphs. Our method is built on earlier ones used in [21,22,18]. However, instead of utilizing the well-known lemma of Bondy and Simonovits [4] that most existing methods do, we develop a new and very simple lemma in its place. One useful feature of the new lemma is its adaptiveness for the hypergraph setting.

Using this new method, we prove a conjecture of Verstraëte [38] that for $r \geq 3$, every r -uniform hypergraph with average degree $\Omega_r(k^{r-1})$ contains Berge cycles of k consecutive lengths. This is sharp up to the constant factor. As a key step and a result of independent interest, we prove that every r -uniform linear hypergraph with average degree at least $7r(k+1)$ contains Berge cycles of k consecutive lengths.

In both of these results, we have additional control on the lengths of the cycles, which therefore also gives us bounds on the Turán numbers of Berge cycles (for even and odd cycles simultaneously). In relation to our main results, we obtain further improvements on the Turán numbers of Berge cycles

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and the Zarankiewicz numbers of even cycles. We will also discuss some potential further applications of our method.

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1. Introduction

The study of cycles is one of the essential ingredients of Graph Theory. In the present paper, we are mainly concerned with extremal problems on cycle lengths in graphs and hypergraphs. One such type of problems consider the set of cycle lengths in graphs under certain density conditions (see, for instance, the work of Sudakov and Verstraete [33] for some in-depth discussions). Another type of problems, which are most pertinent to this paper, consider the longest possible consecutive sequence of cycle lengths.

This can be traced back to a question of Erdős and now a theorem of Bondy and Vince [5] that any graph with minimum degree at least three contains two cycles whose lengths differ by at most two. Häggkvist and Scott [23] extended this by showing that every graph with minimum degree $\Omega(k^2)$ contains k cycles of consecutive even lengths. This quadratic bound was first improved to a linear one by Verstraëte in [37], who proved that average degree at least $8k$ will ensure the existence of k cycles of consecutive even lengths. Since then there has been an extensive research [15,33,27,29] on related topics. Very recently Liu and the second author [28] proved a tight result that every graph G with minimum degree at least $2k + 1$ contains k cycles of consecutive even lengths; and if G is 2-connected and non-bipartite, then G also contains k cycles of consecutive odd lengths. Among others, one closely related problem is the study of cycle lengths modulo a fixed integer k . This was proposed by Burr and Erdős [9] forty years ago and some conjectures were formalized by Thomassen [35] in 1983 (we refer interested readers to [34] for a thorough introduction).

Our work is also closely related to the so-called *Turán problem*. Let \mathcal{F} be a family of r -graphs. An r -graph is \mathcal{F} -free if it does not contain any member of \mathcal{F} as a subhypergraph. The *Turán number* $ex_r(n, \mathcal{F})$ of the family \mathcal{F} denotes the maximum number of hyperedges contained in an n -vertex \mathcal{F} -free r -graph. When $r = 2$, we will write it as $ex(n, \mathcal{F})$. Studying the Turán function $ex_r(n, \mathcal{F})$ for graphs and hypergraphs has been a central problem in extremal graph theory ever since the work of P. Turán [36]. For non-bipartite graphs, the problem is asymptotically solved by the celebrated Erdős–Stone–Simonovits Theorem [14] (see also [13]). However, the Turán problem for bipartite graphs remains mostly open, with the special case for even cycles C_{2k} receiving particular attention. A classic theorem of Bondy and Simonovits [4] shows that $ex(n, C_{2k}) \leq 100k \cdot n^{1+1/k}$. This bound was subsequently improved by several authors in [37,31,6]. The Turán problem for cycles in hypergraphs has been investigated for different notions of hypergraph cycles in [3,17,18,20,21,26] among others. Our work in this paper focuses on so-called *Berge cycles*. The method we develop here also works for some other common notions of

cycles, such as so-called *linear cycles* (or sometimes known as *loose cycles*). See Section 6 for discussion in that direction.

A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ consists of a set \mathcal{V} of *vertices* and a collection \mathcal{E} of subsets of \mathcal{V} . We call a member of \mathcal{E} a *hyperedge* or simply an *edge* of \mathcal{H} . A hypergraph \mathcal{H} is *r-uniform* if all of its edges are *r*-subsets of $\mathcal{V}(H)$. We also simply call an *r*-uniform hypergraph an *r-graph* for brevity. A *Berge path of length ℓ* is a hypergraph \mathcal{P} consisting of ℓ distinct edges e_1, \dots, e_ℓ such that there exist $\ell + 1$ distinct vertices $v_1, \dots, v_{\ell+1}$ satisfying that $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, \ell$. We call the 2-uniform path $v_1v_2 \dots v_{\ell+1}$ a *spine* of the Berge path \mathcal{P} . A *Berge cycle of length ℓ* is a hypergraph \mathcal{C} consisting of ℓ distinct hyperedges e_1, \dots, e_ℓ such that there exist ℓ distinct vertices v_1, \dots, v_ℓ satisfying that $v_i, v_{i+1} \in e_i$ for each $i = 1, \dots, \ell - 1$ and $v_1, v_\ell \in e_\ell$. We call the 2-uniform cycle $v_1v_2 \dots v_\ell v_1$ a *spine* of the Berge cycle \mathcal{C} .

Let us now recall Verstraëte’s theorem on cycles of consecutive even lengths in graphs.

Theorem 1.1. (Verstraëte [37]) *Let $k \geq 2$ be a natural number and G a bipartite graph of average degree at least $4k$ and radius h . Then G contains cycles of k consecutive even lengths. Moreover, the shortest of these cycles has length at most $2h$.*

Let us note that Verstraëte’s result, together with a simple induction, immediately yields $ex(n, C_{2k}) \leq 8kn^{1+1/k}$, thereby giving a short proof of the theorem of Bondy and Simonovits [4] along with an improved coefficient. In an attempt to generalize Theorem 1.1, Verstraëte [38] made the following conjecture for Berge cycles in *r*-graphs.

Conjecture 1.2. (Verstraëte [38]) *Let $r \geq 3$. If \mathcal{H} is an *r*-graph which does not contain Berge cycles of k consecutive lengths, then \mathcal{H} has average degree $O_r(k^{r-1})$ as $k \rightarrow \infty$.*

The complete *r*-graph on k vertices shows that this conjecture, if true, is best possible up to some constant factor (depending only on *r*).

Our main result is to confirm Conjecture 1.2 by showing

Theorem 1.3. *Let $r \geq 3$. Any *r*-graph \mathcal{H} with average degree at least $35rk^{r-1} + 21rk^{r-2}$ contains Berge cycles of k consecutive lengths.*

Moreover, we will establish a stronger statement, in which, as in Theorem 1.1, we have additional control on the length of the shortest cycle in the collection, in terms of a parameter of a hypergraph that is related to the radius of a graph. This stronger version then immediately yields bounds on the Turán numbers of Berge cycles (for both even and odd cycles simultaneously). See Section 5 for detailed discussions.

A hypergraph \mathcal{H} is *linear*, if $|e \cap f| \leq 1$ for any distinct hyperedges $e, f \in \mathcal{E}(\mathcal{H})$. For linear hypergraphs, we have the following even stronger result.

Theorem 1.4. *Let $r \geq 3$. Any linear *r*-graph \mathcal{H} with average degree at least $7r(k + 1)$ contains Berge cycles of k consecutive lengths.*

This is also tight up to the constant factor, by considering a Steiner triple system on k vertices. As with Theorem 1.3, we will also establish a stronger version of Theorem 1.4 with control on the length of the shortest cycle in the collection.

As we shall see, Theorem 1.4 plays a central role in our results. In fact, we will prove Theorem 1.3 in Section 4 by reducing it to the $r = 3$ case of Theorem 1.4 via some reduction lemmas. These reduction lemmas given in Section 4 may be of independent interest in the study of Turán numbers of other Berge hypergraphs.

Let us now say a few words about the method we use in this paper, which we feel should find many future applications. In many Turán type results on cycles in graphs, the following lemma plays an important role. The lemma was implicit in Bondy and Simonovits [4] and was explicit in Verstraëte [37]. Let G be a graph and A, B be disjoint subsets of $V(G)$. An (A, B) -path is a path that has one endpoint in A and the other in B .

Lemma 1.5. (The (A, B) -path lemma) ([4], see also [37]) *Let H be a graph comprising a cycle with a chord. Let (A, B) be a nontrivial partition of $V(H)$. Then H contains (A, B) -paths of every length less than $|V(H)|$, unless H is bipartite with bipartition (A, B) .*

Despite its highly successful applications, one restriction of this powerful lemma is, however, that it does not easily generalize to hypergraphs, especially for cycle structures that are more restrictive than Berge cycles. Faudree and Simonovits [16] introduced an alternative method, called the blowup method. They used the method to study the Turán number of a theta graph (which is the union of a number of internally disjoint paths of the same length between two fixed vertices). The method of Faudree and Simonovits allows for applications to hypergraphs more easily (see [7] for such an application). However, the constant involved in the Faudree–Simonovits method is usually very large. Therefore, it is desirable to look for new methods to add to the existing ones.

The method we use in this paper builds on the ones used in [21,22,18], but contains some novel ingredients and overcomes some drawbacks of the existing methods. The key ideas of our method are contained in the proofs of Lemma 3.1 and Lemma 3.2, both of which rely on a simple lemma (Lemma 2.6 and its hypergraph extension), used in place of the (A, B) -path lemma of Bondy and Simonovits. We expect our method to find further applications in both the graph and hypergraph settings.

The rest of the paper is organized as follows. In Section 2, we introduce notation, some lemmas, and a generalization of the breadth-first search tree for 3-graphs. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.3. In Section 5, we discuss the Turán numbers of Berge cycles in r -graphs and the Zarankiewicz numbers of even cycles. In Section 6, we discuss some future directions. Throughout this paper we make some but not significant efforts to optimize the constant factors used in the proofs.

2. Preliminaries

2.1. Notation and terminology

Unless otherwise specified, all hypergraphs \mathcal{H} discussed in this paper are *simple*, that is, all edges of \mathcal{H} are distinct subsets. The k -*shadow* $\partial_k(\mathcal{H})$ of \mathcal{H} denotes the k -graph on the same vertex set of \mathcal{H} , whose edge set consists of all subsets of k vertices contained in members of $\mathcal{E}(\mathcal{H})$. If $k = 2$, we then just write it as $\partial\mathcal{H}$ and call it *shadow* of \mathcal{H} . We say a subhypergraph \mathcal{G} of $\partial_k(\mathcal{H})$ is *extendable* in \mathcal{H} , if there exists an injection $\psi : \mathcal{E}(\mathcal{G}) \rightarrow \mathcal{E}(\mathcal{H})$ such that $e \subseteq \psi(e)$ for each $e \in \mathcal{E}(\mathcal{G})$. Such an injection ψ is called an *extension* of \mathcal{G} . It is immediate from our definition that

\mathcal{H} contains a Berge cycle of length $\ell \iff \partial\mathcal{H}$ contains an extendable cycle C_ℓ .

Let \mathcal{H} be a hypergraph. By $|\mathcal{H}|$, we mean the total number of hyperedges in \mathcal{H} . For a vertex $v \in \mathcal{V}(\mathcal{H})$, the *degree* $d_{\mathcal{H}}(v)$ denotes the number of hyperedges in \mathcal{H} containing the vertex v . We use $\delta(\mathcal{H})$ and $d(\mathcal{H})$ to denote the *minimum degree* and the *average degree* of \mathcal{H} , respectively. For a subset $S \subseteq \mathcal{V}(\mathcal{H})$ of size at least two, the *co-degree* $d_{\mathcal{H}}(S)$ denotes the number of hyperedges of \mathcal{H} containing S . If $S = \{u, v\}$, then we will just write it as $d_{\mathcal{H}}(u, v)$. For $k \geq 2$, the *minimum k -degree* $\delta_k(\mathcal{H})$ of \mathcal{H} is the minimum of non-zero co-degrees $d_{\mathcal{H}}(S)$ over all subsets $S \subseteq \mathcal{V}(\mathcal{H})$ of size k . A subset $S \subseteq \mathcal{V}(\mathcal{H})$ is a *vertex cover* of \mathcal{H} if each edge of \mathcal{H} contains a vertex in S .

A *linear path* of length ℓ is a hypergraph with edges e_1, e_2, \dots, e_ℓ such that $|e_i \cap e_{i+1}| = 1$ for every $1 \leq i \leq \ell - 1$, and $e_i \cap e_j = \emptyset$ for other pairs $\{i, j\}$, where $i \neq j$. A vertex in the first or last edge of a linear path \mathcal{P} that has degree 1 in \mathcal{P} is called an *endpoint* of \mathcal{P} . A *linear cycle* of length ℓ is a hypergraph with edges e_1, e_2, \dots, e_ℓ such that $|e_i \cap e_{i+1}| = 1$ for every $1 \leq i \leq \ell - 1$, $|e_1 \cap e_\ell| = 1$, and $e_i \cap e_j = \emptyset$ for other pairs $\{i, j\}$, where $i \neq j$. Certainly a linear path/cycle is also a Berge path/cycle.³

2.2. Some lemmas on paths and cycles in hypergraphs

In this subsection, we collect and establish several lemmas to be used later, including the pivotal Lemma 2.6 and its hypergraph extension as mentioned in the introduction.

An r -graph \mathcal{H} is *r -partite*, if there exists an r -partition (A_1, A_2, \dots, A_r) of $\mathcal{V}(\mathcal{H})$ such that for any edge $e \in \mathcal{E}(\mathcal{H})$, $|e \cap A_i| = 1$ for each $i \in [r]$. The following lemma from [11] is well-known.

Lemma 2.1. ([11]) *Every r -graph \mathcal{H} has an r -partite subhypergraph \mathcal{H}' satisfying $|\mathcal{H}'| \geq \frac{r!}{r^r} \cdot |\mathcal{H}|$.*

³ The *length* of a path or a cycle (of any kind) always denotes the number of edges it contains.

Lemma 2.2. *Every r -graph \mathcal{H} has a subhypergraph \mathcal{H}' satisfying that $d(\mathcal{H}') \geq d(\mathcal{H})$ and $\delta(\mathcal{H}') \geq \frac{1}{r} \cdot d(\mathcal{H})$.*

Proof. Let $d = d(\mathcal{H})$. Then $|\mathcal{H}| = |\mathcal{V}(\mathcal{H})| \cdot d/r$. Let \mathcal{H}' be a smallest non-empty subhypergraph of \mathcal{H} satisfying $|\mathcal{H}'| \geq |\mathcal{V}(\mathcal{H}')| \cdot d/r$; such an \mathcal{H}' exists since \mathcal{H} itself satisfies the inequality, and it is clear that \mathcal{H}' has at least r vertices. If \mathcal{H}' contains a vertex x of degree less than d/r , then deleting x from \mathcal{H}' yields a smaller subhypergraph \mathcal{H}'' with $|\mathcal{H}''| \geq |\mathcal{V}(\mathcal{H}'')| \cdot d/r$, a contradiction. Hence $\delta(\mathcal{H}') \geq d/r$. Also, since $|\mathcal{H}'| \geq |\mathcal{V}(\mathcal{H}')| \cdot d/r$, we have $d(\mathcal{H}') \geq d$. \square

We need the following classic result of Erdős and Gallai [10].

Theorem 2.3. [10] *Every n -vertex graph with more than $m(n - 1)/2$ edges has a cycle of length at least $m + 1$. Every n -vertex graph with more than $mn/2$ edges has a path of length at least $m + 1$.*

Lemma 2.4. *Let \mathcal{H} be a linear 3-partite 3-graph with a 3-partition (V_1, V_2, V_3) such that $d(\mathcal{H}) \geq \frac{3p}{2}$. There exists a vertex $u \in V_1$ such that for each $\ell \in [p]$, there exists an extendable path of length ℓ in $\partial\mathcal{H}$ from u to a vertex in V_2 .*

Proof. Let $n = |\mathcal{V}(\mathcal{H})|$. Let $G = \{(e \cap (V_1 \cup V_3)) : e \in \mathcal{H}\}$. Since \mathcal{H} is linear, $|G| = |\mathcal{H}| \geq \frac{pn}{2} > \frac{pn(G)}{2}$. Let $t = \lceil \frac{p}{2} \rceil$. Then $2t \leq p + 1$. Theorem 2.3, G contains a path P of length $2t$. Let $P = u_1v_1u_2v_2 \dots u_tv_t$. By symmetry, we may assume that $u_1 \in V_1$. Then $\forall i \in [s], u_i \in V_1, v_i \in V_3$. Since \mathcal{H} is 3-partite and linear, different edges of P extend to different edges of \mathcal{H} .

For $s \in [t]$, we construct extendable paths of length $2s - 1$ and $2s$, respectively, in $\partial\mathcal{H}$ from u_1 to a vertex in V_2 . Let $u_s v_s w$ be the unique edge of \mathcal{H} containing $u_s v_s$, where $w \in V_2$. Let $P_{2s-1} = u_1 v_1 \dots u_s w$. Then P_{2s-1} has length $2s - 1$ and it is easy to see that it is extendable. Next, let $v_s u_{s+1} w'$ be the unique edge of \mathcal{H} containing $v_s u_{s+1}$, where $w' \in V_2$. Let $P_{2s} = u_1 v_1 \dots u_s v_s w'$. Then P_{2s} has length $2s$ and is extendable, completing the proof. \square

We say that a hypergraph \mathcal{H} is *connected* if $\partial\mathcal{H}$ is connected.

Lemma 2.5. *Let \mathcal{H} be a connected linear hypergraph and X, Y two disjoint sets of vertices in \mathcal{H} . Then there exists a linear (X, Y) -path in \mathcal{H} .*

Proof. Since \mathcal{H} is connected, $\partial\mathcal{H}$ is a connected 2-graph. Let P be a shortest (X, Y) -path in $\partial\mathcal{H}$. Suppose $P = u_1 u_2 \dots u_m$ where $u_1 \in X, u_m \in Y$. For each $i \in [m - 1]$, let e_i denote the unique edge of \mathcal{H} containing $u_i u_{i+1}$. If $e_i \cap V(P) \neq \{u_i, u_{i+1}\}$ for some $i \in [m - 1]$, then we can find a shorter (X, Y) -path in $\partial\mathcal{H}$ than P , a contradiction. Likewise if $e_i \cap e_j$ contains a vertex outside $V(P)$ then since \mathcal{H} is linear, $|i - j| \geq 2$, in which case we can

find a shorter (X, Y) -path in $\partial\mathcal{H}$ than P , a contradiction. Therefore, e_1, e_2, \dots, e_m form a linear (X, Y) -path in \mathcal{H} . \square

The next lemma can be viewed as a replacement for the (A, B) -path lemma (i.e., Lemma 1.5). This lemma and its hypergraph extension will be crucial in our proofs.⁴

Lemma 2.6. *Let G be a connected graph whose edges are colored with 1 and 2 such that there is at least an edge of each color. For $i \in [2]$, let G_i denote the subgraph of G consisting of edges of color i . If $d(G_1) \geq p + 1$, then there exists a path of length at least p in G such that the first edge has color 2 and the others all have color 1.*

Proof. Let $n = |V(G)|$. By our assumption, $e(G_1) \geq (p+1)n/2$ and there is an edge xy of color 2. Let $G'_1 = G_1 - x$. Then $e(G'_1) > (p-1)n/2$. By Theorem 2.3, G'_1 contains a cycle C of length at least p . Since G is connected, there exist paths in G from $\{x, y\}$ to $V(C)$. Among them, let P be a shortest one. Let z be the unique vertex in $V(C) \cap V(P)$. Let z' be a neighbor of z on C . If P has none of edges of color 2, then $(P \cup C \cup \{xy\}) \setminus \{zz'\}$ satisfies the requirement. So we may assume that P has an edge of color 2. In this case, let P' be a shortest subpath of P containing z and an edge of color 2. Then $(P' \cup C) \setminus \{zz'\}$ satisfies the requirement. \square

We next give a hypergraph extension of Lemma 2.6. Even though the objects we study are Berge paths and Berge cycles, it appears that developing variants of Lemma 2.6 for linear paths would facilitate the arguments better and may be handy for future study on linear cycles.

Lemma 2.7. (The special-path lemma) *Let $r, p \geq 2$. Let \mathcal{H} be a connected linear r -graph whose edges are colored with 1 and 2 such that there is at least one edge of each color. For $i \in [2]$, let \mathcal{H}_i denote the subhypergraph of \mathcal{H} consisting of edges of color i . Suppose $d(\mathcal{H}_1) \geq r(r-1)(p-1) + 2r$. Then there exists a linear path of length at least p such that the first edge has color 2 and the other edges all have color 1.*

Proof. Let $n = |\mathcal{V}(\mathcal{H})|$. Let h be an edge of color 2. Let u be a vertex in h . Let \mathcal{H}' be obtained from \mathcal{H}_1 by deleting the $r-1$ vertices in h besides u . Since \mathcal{H}_1 is linear, we lose at most $\frac{n-1}{r-1}$ edges by deleting any vertex. Since $|\mathcal{H}_1| \geq (r-1)(p-1)n + 2n$, certainly we have $|\mathcal{H}'| > (r-1)(p-1)n + n$. By Lemma 2.2, \mathcal{H}' contains a subhypergraph \mathcal{H}'' with minimum degree at least $(r-1)(p-1) + 2$. Call a linear path \mathcal{P} in \mathcal{H} good if its first edge has color 2 and the other edges have color 1 and if the last edge contains an endpoint lying in $\mathcal{V}(\mathcal{H}'')$.

First we show that there exists at least one good path in \mathcal{H} . Let $\mathcal{V}(h)$ denote the set of vertices in h . Since \mathcal{H} is connected, by Lemma 2.5, there exist linear paths in \mathcal{H} from

⁴ Note that this lemma yields a better constant than its hypergraph extension.

$\mathcal{V}(h)$ to $\mathcal{V}(\mathcal{H}'')$. Among these paths let \mathcal{P} be a shortest one. Let e denote the unique edge of \mathcal{P} intersecting $\mathcal{V}(h)$ and view it as the first edge of \mathcal{P} and let f denote the unique edge of \mathcal{P} intersecting $\mathcal{V}(\mathcal{H}'')$. It is possible that $e = f$. Let v be a vertex in $f \cap \mathcal{V}(\mathcal{H}'')$. By our choice of \mathcal{P} , v has degree 1 in \mathcal{P} . First suppose that all edges on \mathcal{P} have color 1. By our choice of \mathcal{P} , $\mathcal{Q} = \mathcal{P} \cup h$ is a linear path that starts with a color 2 edge (namely h) but has all color 1 edges otherwise and one of the endpoints of the last edge is in $\mathcal{V}(\mathcal{H}'')$. Hence \mathcal{Q} is a good path. Next, suppose \mathcal{P} contains at least one color 2 edge. In this case, let \mathcal{Q} be the shortest linear-path contained in \mathcal{P} that contains a color 2 edge and the edge f . Then \mathcal{Q} is a good path.

Now, among all good paths in \mathcal{H} , let \mathcal{Q}^* be a longest one. Let e, f denote the first and last edges, respectively. By our assumption, e is the only edge on \mathcal{Q}^* with color 2 and f contains an endpoint z in $\mathcal{V}(\mathcal{H}'')$. Suppose first \mathcal{Q}^* has length at most $p - 1$. Since $\delta(\mathcal{H}'') \geq (r - 1)(p - 1) + 2$, there are at least $(r - 1)(p - 1) + 1$ edges of \mathcal{H}'' besides f that contain z . Since there are at most $(r - 1)(p - 1)$ vertices in $\mathcal{V}(\mathcal{Q}^*) \setminus \{z\}$ and \mathcal{H} is linear, one of these edges e' is disjoint from $\mathcal{V}(\mathcal{Q}^*) \setminus \{z\}$. But then $\mathcal{Q}^* \cup \{e'\}$ is a good path in \mathcal{H} that is longer than \mathcal{Q}^* , a contradiction. Hence \mathcal{Q}^* must have length at least p and it is a linear path that satisfies the claim. \square

2.3. Maximal extendable skeletons

We now define a generalization of the breadth-first search tree for 3-graphs, which was first introduced by Győri and Lemons in [21]. Let \mathcal{H} be a 3-graph and x a vertex of \mathcal{H} . A *maximal extendable skeleton* of \mathcal{H} rooted at x is an extendable subgraph $T \subseteq \partial\mathcal{H}$, obtained by running the following algorithm till its termination.

Algorithm 2.1. (Modified Breadth First Search)

Input: A 3-graph \mathcal{H} and a vertex $x \in V(\mathcal{H})$.

Output: A tree T rooted at x and an extension $\psi : E(T) \rightarrow \mathcal{E}(\mathcal{H})$.

Initiation: Set $T = \emptyset$, $Q = \{x\}$. Label all edges of \mathcal{H} as unprocessed.

Iteration: Maintain Q as a queue. Let u be the first vertex in Q . Consider any unprocessed edge $e = uvw$ containing u . If at least one of v, w is not in $V(T)$, then pick one, say v . Add edge uv to T and add v to the end of Q . Let $\psi(uv) = e$ and mark e as processed and delete e from \mathcal{H} . If both of v, w are already in $V(T)$, then skip edge e and mark it as processed. When all edges containing u have been processed, we delete u from Q .

Termination: Terminate when $Q = \emptyset$. \square

Clearly, the graph T we obtain from Algorithm 2.1 is a tree rooted at x , in which the distance from a vertex y to the root x is non-decreasing in the order in which y is being added to T (or equivalently to the queue Q). We point out that such tree T may not be spanning in $\partial\mathcal{H}$.

Given a 3-graph \mathcal{H} and a maximal extendable skeleton T in \mathcal{H} rooted at some vertex x together with an extension ψ of it, let

$$\mathcal{E}_{T,\psi} = \{\psi(e) : e \in E(T)\} \tag{1}$$

and

$$\mathcal{E}'_{T,\psi} = \{e \in \mathcal{E}(\mathcal{H}) \setminus \mathcal{E}_{T,\psi} : e \cap V(T) \neq \emptyset\}. \tag{2}$$

Note that if \mathcal{H} is a linear 3-graph, then each edge in $\partial\mathcal{H}$ lies in a unique edge of \mathcal{H} . So an extension ψ of a maximal extendable skeleton T in a linear 3-graph \mathcal{H} is uniquely determined by T . In this case, we will simply write \mathcal{E}_T and \mathcal{E}'_T for $\mathcal{E}_{T,\psi}$ and $\mathcal{E}'_{T,\psi}$, respectively.

Given two vertices $u, v \in V(T)$, let $d_T(u, v)$ denote the length of the unique (u, v) -path in T . For each integer $i \geq 0$, let

$$L_i(T) = \{y \in V(T) : d_T(x, y) = i\}. \tag{3}$$

When the context is clear, we will simply write L_i for $L_i(T)$. We call L_i *level i* of T .

Proposition 2.8. *Let \mathcal{H} be a 3-graph and $x \in V(\mathcal{H})$. Let T be a maximal extendable skeleton in \mathcal{H} rooted at x together with an extension ψ . For every $e \in \mathcal{E}'_{T,\psi}$, there exists some $i \geq 0$ such that $e \subseteq L_i(T) \cup L_{i+1}(T)$.*

Proof. We already observed that for any $y, y' \in V(T)$, if y is added to T before y' then $d_T(x, y) \leq d_T(x, y')$. Among all vertices in $e \cap V(T)$, let u be the vertex that is added to T the earliest. Let $i = d_T(x, u)$. Then $u \in L_i(T)$. Suppose $e = uvw$. Consider the time the algorithm processed e . If one of v or w , say v , was not in T at that time, then the algorithm would have added uv to T , deleted e , and let $\psi(uv) = e$, contradicting $e \in \mathcal{E}'_{T,\psi}$. Hence, v, w were both in $V(T)$ at that time. By our earlier discussion, $d_T(x, v), d_T(x, w) \geq i$. Since v was already in T , it must have been added to T when we processed an edge e' containing a vertex u' where u' was either u itself or an earlier vertex in the queue Q . Hence $d_T(x, v) = d_T(x, u') + 1 \leq i + 1$. Similarly, $d_T(x, w) \leq i + 1$. \square

3. Consecutive cycles in linear r -graphs

We devote this section to the proof of Theorem 1.4. The next two lemmas are crucial and their proofs contain the main ideas of our method used in this paper.

Lemma 3.1. *Let \mathcal{H} be a linear 3-graph and T a maximal extendable skeleton in \mathcal{H} rooted at some vertex r . For each $i \geq 1$, let $L_i = \{v : d_T(r, v) = i\}$ and $\mathcal{A}_i = \{e \in \mathcal{H} : |e \cap L_i| = 2, |e \cap L_{i-1}| = 1\}$. If there exists an $i \geq 1$ such that $|\mathcal{A}_i| \geq (k + 2)|L_i|$, then \mathcal{H} contains Berge cycles of all lengths in $[a, a + k - 1]$ for some $a \leq 2i$.*

Proof. Fix an $i \geq 1$ that satisfies $|\mathcal{A}_i| \geq (k + 2)|L_i|$. Since \mathcal{H} is linear, the condition implies that $i \geq 2$. Let $G_i = \{e \cap L_i : e \in \mathcal{A}_i\}$. Since \mathcal{H} is linear, there is a bijection between \mathcal{A}_i and G_i . So, G_i is a 2-graph on L_i with $|G_i| = |\mathcal{A}_i| \geq (k + 2)|L_i|$. Hence, $d(G_i) \geq 2k + 4$. Let G be a connected component of G_i with $d(G) \geq 2k + 4$. By our assumption, each $uv \in E(G)$ has a unique co-neighbor w in \mathcal{A}_i and $w \in L_{i-1}$. Let S be the set of co-neighbors of uv over all edges $uv \in E(G)$. Note that the condition $d(G) \geq 2k + 4$ implies that $|S| \geq 2k + 4 \geq 2$. Indeed, the link of any single vertex can only be a matching.

Let r^* be the closest common ancestor in T of vertices in S and let T^* be unique subtree of T rooted at r^* whose set of leaves is S . Suppose $r^* \in L_{i'}$. Let s_1, \dots, s_m denote the children of r^* in T^* . Color all the descendants of s_1 in T^* that lie in S color 1 and the other vertices in S color 2. Call this coloring c . By our choice of r^* and the fact that $|S| \geq 2$, we have $m \geq 2$. So both color 1 and color 2 appear in S . Now, define an edge-coloring ϕ of G as follows. For each $uv \in E(G)$, let w be the unique vertex in S such that $uvw \in \mathcal{A}_i$. Let $\phi(uv) = c(w)$. This defines a 2-coloring of the edges of G in which there is at least one edge of each color. For $i \in [2]$, let G_i denote the subgraph of G consisting of edges of color i . We may assume that $e(G_1) \geq e(G_2)$ since the arguments for the $e(G_2) \geq e(G_1)$ case are identical. By our assumption, $d(G_1) \geq k + 2$. Since G is connected, by Lemma 2.6, there exists a path P of length $k + 1$ in G such that first edge has color 2 and all the others have color 1. Suppose $P = uv_1v_2 \dots v_{k+1}$, where uv_1 is the only edge of color 2. Let x be the unique vertex in S such that $uv_1x \in \mathcal{A}_i$. Let X denote the unique path in T^* from r^* to x . For each $\ell \in [k]$, let y_ℓ be the unique vertex in S such that $v_\ell v_{\ell+1}y_\ell \in \mathcal{A}_i$. Let Y_ℓ denote the unique path in T^* from r^* to y_ℓ .

Now, $P_\ell = xv_1 \dots v_\ell y_\ell$ is an extendable path of length $\ell + 1$ whose edges extend to different hyperedges of $\mathcal{A}_i \subseteq \mathcal{E}'_T$. Clearly $X \cup Y_\ell$ is an extendable path of length $2(i - 1 - i')$ whose edges extend to different hyperedges in \mathcal{E}_T . Further, $V(X \cup Y_\ell) \cap V(P_\ell) = \{x, y_\ell\}$. Hence, $P_\ell \cup X \cup Y_\ell$ is an extendable cycle in $\partial(\mathcal{H})$ of length $2(i - 1 - i') + \ell + 1$, where $\ell \in [k]$. So \mathcal{H} contains a Berge cycle of each length in $[a, a + k - 1]$, where $a = 2(i - i') \leq 2i$. \square

Lemma 3.2. *Let \mathcal{H} be a linear 3-graph and T a maximal extendable skeleton in \mathcal{H} rooted at some vertex r . For each $i \geq 1$, let $L_i = \{v : d_T(r, v) = i\}$, $\mathcal{B}_i = \{e \in \mathcal{H} : e \subseteq L_i\}$, and $\mathcal{C}_i = \{e \in \mathcal{H} : |e \cap L_i| = 2, |e \cap L_{i+1}| = 1\}$. For each $i \geq 1$, if there is no $a \leq 2i + 2$ such that \mathcal{H} contains Berge cycles of all lengths in $[a, a + k - 1]$, then $|\mathcal{B}_i \cup \mathcal{C}_i| \leq (\frac{7k}{2} + 1)|L_i| + (\frac{5k}{2} + 2)|L_{i+1}|$.*

Proof. Fix an i for which \mathcal{H} does not contains Berge cycles of all lengths in $[a, a + k - 1]$ for any $a \leq 2i + 2$. Let \mathcal{G} be a connected nontrivial component of $\mathcal{B}_i \cup \mathcal{C}_i$. Let $n_i = |\mathcal{V}(\mathcal{G}) \cap L_i|$ and $n_{i+1} = |\mathcal{V}(\mathcal{G}) \cap L_{i+1}|$. Our goal is to show $|\mathcal{G}| \leq (\frac{7k}{2} + 1)n_i + (\frac{5k}{2} + 2)n_{i+1}$.

Let r^* denote the closest common ancestor of $\mathcal{V}(\mathcal{G})$ in T . Suppose $r^* \in L_{i'}$. Let T^* be the minimal subtree of T rooted at r^* which contains $\mathcal{V}(\mathcal{G})$. Let s_1, \dots, s_m denote the children of r^* in T^* . By our choice of r^* , $m \geq 2$. Now, color all descendants of s_1

in $V(T^*)$ with color 1 and the descendants of other s_j 's in $V(T^*)$ with color 2. Call this coloring c . By the minimality of T , we see that \mathcal{G} has vertices of both colors.

In the following Claim, we present one ingredient that we will repeatedly use.

Claim 1. Suppose $\partial\mathcal{G}$ contains an extendable path P of length ℓ with endpoints x, y such that $c(x) \neq c(y)$ and $V(X \cup Y) \cap V(P) = \{x, y\}$, where X, Y denote the unique paths in T^* from r^* to x, y , respectively. Then $X \cup Y \cup P$ is an extendable cycle of length $(i(x) + i(y) - 2i') + \ell$, where $i(x), i(y)$ denote the levels x, y are in, respectively.

Proof. Since $c(x) \neq c(y)$, $V(X) \cap V(Y) = \{r^*\}$. Thus $C = X \cup Y \cup P$ is a cycle in $\partial\mathcal{H}$ of length $|X| + |Y| + \ell$. By our assumption different edges on $X \cup Y$ extend to different hyperedges of \mathcal{E}_T , while different edges on P extend to different hyperedges of \mathcal{E}'_T . Hence C is an extendable cycle in $\partial\mathcal{H}$. Lastly, note that $|X| = i(x) - i'$ and $|Y| = i(y) - i'$. \square

For each edge $e \in \mathcal{G}$, we call it *monochromatic* if all three vertices in e have the same color in c ; otherwise we call it *non-monochromatic*. Let \mathcal{M} denote the subgraph of \mathcal{G} consisting of monochromatic edges and \mathcal{N} the subgraph of \mathcal{G} consisting of non-monochromatic edges. So $\mathcal{G} = \mathcal{M} \cup \mathcal{N}$, and by our choice of r^* , $\mathcal{N} \neq \emptyset$.

Claim 2. $|\mathcal{M}| \leq (2k + 2)(n_i + n_{i+1})$.

Proof. Suppose that $|\mathcal{M}| > (2k + 2)(n_i + n_{i+1})$. We will derive a contradiction. By Lemma 2.7, there exists a linear path \mathcal{P} in \mathcal{G} of length at least $k + 1$ such that the first edge is in \mathcal{N} and all the other edges are in \mathcal{M} . Let the edges of \mathcal{P} be e_1, e_2, \dots, e_{k+1} in order where e_1 is the edge in \mathcal{N} . It is easy to see that all the vertices in $\mathcal{V}(\mathcal{P} \setminus e_1)$ must have the same color under c . By renaming the colors in c if necessary, we may assume that they have color 1. Also, among the two endpoints x, x' of \mathcal{P} contained in e_1 , either $c(x) \neq 1$ or $c(x') \neq 1$. Without loss of generality, suppose that $c(x) = 2$. For each $\ell \in [k + 1]$, let \mathcal{P}_ℓ denote the linear path consisting of e_1, \dots, e_ℓ . By our definition of \mathcal{G} , $x \in L_i \cup L_{i+1}$.

First suppose that $x \in L_{i+1}$. Let y_1 be any vertex in e_1 that has color 1 and lies in L_i (note that such a vertex exists). Let $\ell \in \{2, \dots, k\}$. By our discussion, all three vertices in e_ℓ have color 1, and among the two endpoints in e_ℓ of \mathcal{P}_ℓ , there exists one that lies in L_i , denote that vertex by y_ℓ . Now \mathcal{P}_ℓ is a linear (x, y_ℓ) -path of length ℓ in \mathcal{G} whose vertices are contained in $L_i \cup L_{i+1}$. Let P_ℓ denote a spine of it with x, y_ℓ as endpoints. Note that all vertices on P_ℓ except x have color 1 in c and $c(x) = 2$. Let X, Y_ℓ denote the unique paths in T^* from r^* to x and y_ℓ , respectively. By the definition of Y_ℓ , Y_ℓ intersects P_ℓ only at y_ℓ . Also, X contains a vertex each from L_i and L_{i+1} , both of which have color 2 in c , but all vertices on P_ℓ other than x have color 1. Hence X intersects P_ℓ only at x . By Claim 1, $X \cup Y_\ell \cup P_\ell$ is an extendable cycle of length $(i - i') + (i + 1 - i') + \ell$ in $\partial\mathcal{H}$. Since this holds for each $\ell \in [k]$, we get Berge cycles of all lengths in $[2(i - i') + 2, 2(i - i') + k + 1]$. This contradicts our assumption that \mathcal{H} has no Berge cycles of k consecutive lengths with the shortest having length at most $2i + 2$.

Next, suppose $x \in L_i$. Let $\ell \in \{2, \dots, k + 1\}$. By definition all the vertices in e_ℓ have color 1 under c . Among the two endpoints in e_ℓ of \mathcal{P}_ℓ , one of them lies in L_i . Let y_ℓ denote such a vertex. By a similar argument as above, we can obtain a Berge cycle of length $(i - i') + (i - i') + \ell$ for each $\ell \in \{2, \dots, k + 1\}$. Hence \mathcal{H} has Berge cycles of all lengths in $[2(i - i') + 2, 2(i - i') + k + 1]$. Again, this contradicts our assumption that \mathcal{H} has no Berge cycles of k consecutive lengths the shortest of which has length at most $2i + 2$. This proves Claim 2. \square

Claim 3. $|\mathcal{N}| \leq (\frac{3k}{2} - 1)n_i + \frac{k}{2}n_{i+1}$.

Proof. We further decompose \mathcal{N} as follows. Let S_1, S_2 denote the set of descendants of r^* in L_i with color 1 and 2, respectively. Let

$$\begin{aligned} \mathcal{N}_1 &= \{e \in \mathcal{N} : |e \cap S_1| = 2\} \\ \mathcal{N}_2 &= \{e \in \mathcal{N} : |e \cap S_2| = 2\} \\ \mathcal{N}_3 &= \{e \in \mathcal{N} \cap \mathcal{C}_i : |e \cap S_1| = |e \cap S_2| = 1\} \end{aligned}$$

By definition, any $e \in \mathcal{N} \cap \mathcal{B}_i$ intersects both S_1 and S_2 . Thus we see that $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. Suppose first that $|\mathcal{N}_1| > \frac{k-1}{2}n_i$. Let G_1 be formed by taking the pair of color 1 vertices from each hyperedge in \mathcal{N}_1 . Then $|G_1| = |\mathcal{N}_1| > \frac{k-1}{2}n_i > \frac{k-1}{2}n(G_1)$. By Theorem 2.3, G_1 contains a path P of length at least k . Let $P = y_0y_1 \dots y_k$. Let xy_0y_1 be the unique hyperedge in \mathcal{N}_1 containing y_0y_1 . Then $c(x) = 2$ and $x \in L_i \cup L_{i+1}$. Let $p \in \{0, 1\}$ such that $p = 1$ iff $x \in L_{i+1}$. Let X denote the unique path in T^* from r^* to x . For each $\ell \in [k]$, let Y_ℓ denote the unique path in T^* from r^* to y_ℓ . By the definition of \mathcal{N}_1 , the path $P_\ell = xy_1 \dots y_\ell$ is an extendable path of length ℓ in $\partial\mathcal{G}$. Also $V(X) \cap V(P_\ell) = \{x\}$ since x is the only vertex with color 2 on P_ℓ . By definition of Y_ℓ and P_ℓ , $V(Y_\ell) \cap V(P_\ell) = \{y_\ell\}$. So, $V(X \cup Y_\ell) \cap V(P) = \{x, y_\ell\}$. By Claim 1, $X \cup Y_\ell \cup P_\ell$ is an extendable cycle of length $2(i - i') + p + \ell$ in $\partial\mathcal{H}$, where $p \in \{0, 1\}$. Since this holds for each $\ell \in [k]$, \mathcal{H} contains Berge cycles of k consecutive lengths, the shortest of which has length at most $2i + 2$, a contradiction. Hence, $|\mathcal{N}_1| \leq \frac{k-1}{2}n_i$. By symmetry, we have $|\mathcal{N}_2| \leq \frac{k-1}{2}n_i$. Therefore,

$$|\mathcal{N}_1| + |\mathcal{N}_2| \leq (k - 1)n_i \tag{4}$$

Now, consider \mathcal{N}_3 . By definition, \mathcal{N}_3 is 3-partite whose three parts are contained in S_1, S_2 , and L_{i+1} , respectively. Suppose $d(\mathcal{N}_3) > \frac{3k}{2}$. Then by Lemma 2.4, there exists $x \in S_1 \cap \mathcal{V}(\mathcal{N}_3)$ such that for each $\ell \in [k]$, there is an extendable path P_ℓ in $\partial\mathcal{N}_3$ from x to a vertex y_ℓ in $S_2 \cap \mathcal{V}(\mathcal{N}_3)$. Let X denote the unique path in T^* from r^* to x and for each $\ell \in [k]$ let Y_ℓ denote the unique path in T^* from r^* to y_ℓ . Like before, $X \cup Y_\ell \cup P_\ell$ is an extendable cycle of length $2(i - i') + \ell$ in $\partial\mathcal{H}$. Thus \mathcal{H} contains Berge cycles of k consecutive lengths, the shortest of which has length at most $2i$, a contradiction. Hence

$$|\mathcal{N}_3| \leq \frac{k}{2}(n_i + n_{i+1}). \tag{5}$$

Claim 3 now follows from (4) and (5). \square

By Claims 2 and 3, $|\mathcal{G}| \leq (\frac{7k}{2} + 1)n_i + (\frac{5k}{2} + 2)n_{i+1}$. Since this holds for each component of $\mathcal{B}_i \cup \mathcal{C}_i$, it follows $|\mathcal{B}_i \cup \mathcal{C}_i| \leq (\frac{7k}{2} + 1)|L_i| + (\frac{5k}{2} + 2)|L_{i+1}|$. This proves Lemma 3.2. \square

Remark. In the proof of Claim 2 of Lemma 3.2, one could also use Lemma 2.6 instead of Lemma 2.7 and get slightly better constants. However, there are more subtleties to address if one were to use Lemma 2.6 and also we want to demonstrate the use of Lemma 2.7 since the lemma will be useful in the study of linear cycles.

Lemma 3.3. *Let \mathcal{H} be a linear 3-graph. Let T be any maximal extendable skeleton in \mathcal{H} rooted at some vertex r and with height h . If the number of hyperedges in \mathcal{H} that contain some vertex in $V(T)$ is at least $7(k + 1)|V(T)|$ then \mathcal{H} contains Berge cycles of k consecutive lengths, the shortest of which is at most $2h + 2$.*

Proof. Suppose \mathcal{H} does not contain such Berge cycles, we derive a contradiction. Define \mathcal{E}_T and \mathcal{E}'_T as in (1) and (2). For each $i \geq 0$, let $L_i = \{v \in V(T) : d_T(r, v) = i\}$. Let $\mathcal{A}_i = \{e \in \mathcal{E}(\mathcal{H}) : |e \cap L_i| = 2, |L_{i-1}| = 1\}$, $\mathcal{B}_i = \{e \in \mathcal{H} : |e \cap L_i| = 3\}$, and $\mathcal{C}_i = \{e \in \mathcal{H} : |e \cap L_i| = 2, |e \cap L_{i+1}| = 1\}$. By Proposition 2.8, $\mathcal{E}'_T \subseteq \bigcup_{i=1}^h (\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i)$. By Lemma 3.1 and 3.2, for each $i \in [h]$, $|\mathcal{A}_i \cup \mathcal{B}_i \cup \mathcal{C}_i| \leq (\frac{9}{2}k + 3)|L_i| + (\frac{5}{2}k + 2)|L_{i+1}|$. Hence

$$|\mathcal{E}'_T| \leq (\frac{9}{2}k + 3) \sum_{i=1}^h |L_i| + (\frac{5}{2}k + 2) \sum_{i=1}^{h-1} |L_{i+1}| \leq (7k + 5)|V(T)|.$$

On the other hand, $|\mathcal{E}_T| = |V(T)| - 1$. So the number of hyperedges in \mathcal{H} with at least one vertex in $V(T)$ is less than $7(k + 1)|V(T)|$, a contradiction. \square

Remark. Note that in Lemma 3.3, we not only find Berge cycles of k consecutive lengths, but also ensure that the shortest length is no more than twice the height of any maximal extendable skeleton. This extra condition on the shortest length will be useful for studying the Turán number of a Berge cycle of fixed length. See Section 5 for detailed discussions. If we do not impose any condition on the shortest length of our cycles, then we can both improve the bounds and simplify the proofs of Lemma 3.1 and Lemma 3.2.

We now prove the following theorem, which perhaps is the cornerstone of this paper.

Theorem 3.4. *Let \mathcal{H} be a linear 3-graph with $d(\mathcal{H}) \geq 21(k + 1)$. Then \mathcal{H} contains Berge cycles of k consecutive lengths.*

Proof. By our assumption $|\mathcal{H}| \geq 7(k + 1)|\mathcal{V}(\mathcal{H})|$. Let T be any maximal extendable skeleton in \mathcal{H} . If \mathcal{H} does not contain Berge cycles of k consecutive lengths, then by

Lemma 3.3, there are fewer than $7(k + 1)|V(T)|$ hyperedges in \mathcal{H} that contain some vertex in $V(T)$. Let us delete $V(T)$ and hyperedges that contain vertices in $V(T)$. Denote the remaining hypergraph by \mathcal{H}' . We can repeat the process until we either find Berge cycles of k consecutive lengths or we run out of hyperedges. Since $|\mathcal{H}| \geq 7(k + 1)|\mathcal{V}(\mathcal{H})|$ and we lose fewer than $7(k + 1)$ hyperedges per vertex we delete, we never run out of hyperedges. So \mathcal{H} must contain Berge cycles of k consecutive lengths. \square

Now we can derive Theorem 1.4 promptly from Theorem 3.4.

Proof of Theorem 1.4. Let \mathcal{H} be a linear r -graph with average degree at least $7(k + 1)r$. Define \mathcal{G} to be the 3-graph with $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{H})$ by taking as hyperedges a 3-subset from each edge e of \mathcal{H} . Since \mathcal{H} is linear, the 3-subsets are all distinct. So $|\mathcal{G}| = |\mathcal{H}| \geq 7(k + 1)|\mathcal{V}(\mathcal{H})|$. Also, \mathcal{G} is linear. By Theorem 3.4, \mathcal{G} contains Berge cycles of k consecutive lengths. Since all edges of \mathcal{G} extend to distinct edges of \mathcal{H} , these Berge cycles in \mathcal{G} extend to Berge cycles of k consecutive lengths in \mathcal{H} . \square

4. The proof of Theorem 1.3

In this section we prove Theorem 1.3, by essentially reducing it to Theorem 3.4. We start by providing some useful lemmas. The following lemma will be important for our reduction. It may be viewed as a very special case of the delta-system lemma, introduced by Deza, Erdős, and Frankl [8].

Lemma 4.1. *Any r -graph \mathcal{H} has either a subhypergraph \mathcal{H}' with $\delta_{r-1}(\mathcal{H}') \geq k + 1$, or a subhypergraph \mathcal{H}'' with $|\mathcal{H}''| \geq |\mathcal{H}|/k$ in which each hyperedge contains an $(r - 1)$ -set with co-degree 1 in \mathcal{H}'' . In particular, in the latter case, there is an extendable $(r - 1)$ -graph $\mathcal{G} \subseteq \partial_{r-1}(\mathcal{H})$ with $|\mathcal{G}| \geq |\mathcal{H}|/k$.*

Proof. We apply the following greedy algorithm for \mathcal{H} . Initially, set $\mathcal{H}' = \mathcal{H}$ and $\mathcal{H}'' = \emptyset$. If there exist an edge $e \in \mathcal{E}(\mathcal{H}')$ and an $(r - 1)$ -subset $e' \subseteq e$ such that $d_{\mathcal{H}'}(e') \leq k$, then we place one hyperedge containing e' in \mathcal{H}'' and delete all hyperedges in \mathcal{H}' containing e' . We continue this until there is no such pair (e, e') . If \mathcal{H}' is nonempty, then we are done as clearly $\delta_{r-1}(\mathcal{H}') \geq k + 1$. Hence, \mathcal{H}' is empty. Then \mathcal{H}'' satisfies that $|\mathcal{H}''| \geq |\mathcal{H}|/k$ and that each hyperedge e contains an $(r - 1)$ -subset e' with co-degree 1 in \mathcal{H}'' . For the second statement, form \mathcal{G} by selecting e' from each hypergraph e of \mathcal{H}'' . Clearly, $|\mathcal{G}| = |\mathcal{H}''|$ and \mathcal{G} is extendable. \square

In the next two lemmas, we show that if an r -graph \mathcal{H} has large $\delta_{r-1}(\mathcal{H})$, then there exist many Berge cycles of consecutive lengths. An r -graph \mathcal{P} is a *tight path of length m* , if it consists of m edges e_1, \dots, e_m and $m + r - 1$ vertices v_1, \dots, v_{m+r-1} such that each $e_i = \{v_i, \dots, v_{i+r-1}\}$.

Lemma 4.2. *Let $r \geq 3$ and \mathcal{P} be an r -graph. If \mathcal{P} is a tight path of length m , then \mathcal{P} contains Berge cycles of all lengths in $\{3, \dots, m - 1\}$.*

Proof. Let e_1, \dots, e_m be all edges in \mathcal{P} such that each $e_i = \{i, \dots, i + r - 1\}$. Let $f_i = \{i, i + 1, i + 2\}$. Since each f_i extends to e_i , it suffices to find Berge cycles in the tight path $\mathcal{P}' = \{f_1, \dots, f_m\}$ (i.e., it suffices to consider 3-graphs). For even $t \leq m$, consider the following 2-cycle with spine $2, 4, \dots, t, t - 1, t - 3, \dots, 3, 2$. This 2-cycle can extend to a Berge cycle of length $t - 1$ in \mathcal{P}' with edges $f_2, f_4, \dots, f_{t-2}, f_{t-1}, f_{t-3}, \dots, f_3, f_1$. These edges cover pairs $24, 46, \dots, (t - 2)t, t(t - 1), (t - 1)(t - 3), \dots, 53, 32$, respectively. For odd $t \leq m$, similarly consider the following 2-cycle with spine $2, 4, \dots, t - 3, t - 1, t, t - 2, \dots, 3, 2$. This 2-cycle can extend to a Berge cycle of length $t - 1$ in \mathcal{P}' with edges $f_2, f_4, \dots, f_{t-3}, f_{t-1}, f_{t-2}, f_{t-4}, \dots, f_3, f_1$. These edges cover pairs $24, 46, \dots, (t - 3)(t - 1), (t - 1)t, t(t - 2), (t - 2)(t - 4), \dots, 53, 32$, respectively. Hence, there exist Berge cycles of all lengths in $\{3, \dots, m - 1\}$ in \mathcal{P}' (and thus in \mathcal{P}). \square

Lemma 4.3. *Let $r \geq 3$ and \mathcal{H} be an r -graph with $\delta_{r-1}(\mathcal{H}) \geq k + 1$. Then \mathcal{H} contains Berge cycles of all lengths in $\{3, 4, \dots, k + 2\}$.*

Proof. Let \mathcal{P} be a longest tight path in \mathcal{H} , say of length m . Let e_1, \dots, e_m be all edges in \mathcal{P} such that each $e_i = \{v_i, \dots, v_{i+r-1}\}$. Let $S = \{v_{m+1}, \dots, v_{m+r-1}\}$. All edges f of \mathcal{H} containing S satisfy that $f \setminus S \subseteq \{v_1, \dots, v_m\}$, as otherwise $\mathcal{P} \cup f$ would be a longer tight path than \mathcal{P} , contradicting the choice of \mathcal{P} . This also shows that $m \geq d_{\mathcal{H}}(S) \geq \delta_{r-1}(\mathcal{H}) \geq k + 1$. By Lemma 4.2, \mathcal{P} contains Berge cycles of all lengths in $\{3, \dots, m - 1\}$. Hence, we may assume that $k + 1 \leq m \leq k + 2$ and in particular there exist Berge cycles of all lengths in $\{3, 4, \dots, k\}$ in \mathcal{H} .

Suppose $m = k + 1$. It is clear that $d_{\mathcal{H}}(S) = k + 1$ and all $k + 1$ edges f_1, \dots, f_{k+1} in \mathcal{H} containing S are such that $f_i = S \cup \{v_i\}$. Then there exist a Berge cycle $\{e_1, \dots, e_{k+1}, f_1\}$ of length $k + 2$ with spine $v_1, v_2, \dots, v_{k+1}, v_{k+2}$ and a Berge cycle $\{e_2, \dots, e_{k+1}, f_2\}$ of length $k + 1$ with spine $v_2, \dots, v_{k+1}, v_{k+2}$. So \mathcal{H} contains Berge cycles of all lengths in $\{3, 4, \dots, k + 2\}$.

Therefore, we have $m = k + 2$. There are at least $k + 1$ edges f_1, \dots, f_{k+1} in \mathcal{H} containing S such that $f_i \setminus S \subseteq \{v_1, \dots, v_{k+2}\}$. So there is some $i \in [k + 2]$ such that for every $j \in [k + 2] \setminus \{i\}$, $S \cup \{v_j\} \in \mathcal{E}(\mathcal{H})$. Let j_0, j_1 be the first and second integers in $[k + 2] \setminus \{i\}$. Then there exist a Berge cycle $\{e_j : j \in [k + 2] \setminus \{i\}\} \cup (S \cup \{v_{j_0}\})$ of length $k + 2$ with spine $\{v_j : j \in [k + 3] \setminus \{i\}\}$, and a Berge cycle $\{e_j : j \in [k + 2] \setminus \{i, j_0\}\} \cup (S \cup \{v_{j_1}\})$ of length $k + 1$ with spine $\{v_j : j \in [k + 3] \setminus \{i, j_0\}\}$. This finishes the proof. \square

We are now ready to prove Theorem 1.3. We will use induction on r . The following theorem forms the basis step.

Theorem 4.4. *Let \mathcal{G} be a 3-graph with $d(\mathcal{G}) \geq 105k^2 + 63k$. Then \mathcal{G} contains Berge cycles of k consecutive lengths.*

Proof. By Lemma 4.1, \mathcal{G} has either a subhypergraph \mathcal{G}' with $\delta_2(\mathcal{G}') \geq k + 1$, or a subhypergraph \mathcal{G}'' with $|\mathcal{G}''| \geq |\mathcal{G}|/k$ in which each hyperedge contains a pair that has co-degree 1 in \mathcal{G}'' . In the former case, by Lemma 4.3, \mathcal{G}' contains Berge cycles of k consecutive lengths and we are done. Hence, we may assume the latter case. For each hyperedge in \mathcal{G}'' let us mark a pair in it that has co-degree 1. By our assumption, each hyperedge in \mathcal{G}'' has a marked pair.

Let us call a pair uv a *high pair* if its co-degree in \mathcal{G}'' is at least 3 and a *low pair* otherwise. Let \mathcal{G}_1 consist of all the hyperedges in \mathcal{G}'' that contain a high pair and \mathcal{G}_2 consist of all the other hyperedges in \mathcal{G}'' . Since $d(\mathcal{G}'') \geq d(\mathcal{G})/k \geq 105k + 63$, one of the following two cases applies.

Case 1. $d(\mathcal{G}_1) \geq 42k$.

Let S be a random set of vertices with each vertex of \mathcal{G}_1 selected independently with probability $\frac{2}{3}$. For each hyperedge in \mathcal{G}_1 , call it *good* for S if the two vertices in its marked pair are both in S and the third vertex is not in S . The probability of a hyperedge being good is $\frac{4}{27}$. So there exists a set S for which at least $\frac{4}{27}|\mathcal{G}_1|$ of the hyperedges are good. Fix such a set S and let \mathcal{G}_1^* consist of all the good hyperedges of \mathcal{G}_1 . By our assumption

$$d(\mathcal{G}_1^*) \geq \frac{4}{27}d(\mathcal{G}_1) \geq 6k.$$

Let $G_1 = \{e \cap S : e \in \mathcal{G}_1^*\}$. Then G_1 is a 2-graph and there is a bijection between edges in G_1 and hyperedges in \mathcal{G}_1^* . In particular, $|G_1| = |\mathcal{G}_1^*|$. Hence $d(G_1) \geq \frac{2}{3}d(\mathcal{G}_1^*) \geq 4k$. Since G_1 contains a bipartite subgraph with average degree at least $\frac{1}{2}d(G_1) \geq 2k$, by Theorem 1.1, G_1 contains cycles of $k/2$ consecutive even lengths. To complete this case, observe that if C is a cycle of length ℓ in G_1 , then \mathcal{G} contains a Berge cycle of length ℓ and a Berge cycle of length $\ell + 1$. Indeed, suppose $C = u_1u_2 \dots u_\ell u_1$. By our definition, all edges on C extend to different hyperedges in \mathcal{G} . So we obtain a Berge cycle of length ℓ . Let u_1u_2w be the unique hyperedge in \mathcal{G}_1^* containing $\{u_1, u_2\}$. By definition, $w \notin S$. Also, at least one pair in u_1u_2w is a high pair. Since u_1u_2 is a marked pair and has co-degree 1, either u_1w or u_2w is a high pair. By symmetry suppose u_1w is a high pair. Since u_1w has co-degree at least 3 in \mathcal{G}'' there is a hyperedge u_1wz where $z \notin \{u_2, u_\ell\}$. It is easy to see that $u_1wu_2 \dots u_\ell u_1$ is an extendable cycle in $\partial(\mathcal{G}'')$ of length $\ell + 1$. So \mathcal{G} contains a Berge cycle of length $\ell + 1$.

Case 2. $d(\mathcal{G}_2) \geq 63(k + 1)$.

By our assumption, for each hyperedge in \mathcal{G}_2 , one of its pairs have co-degree 1 and the other two have co-degree at most 2. Define an auxiliary graph L whose vertices are hyperedges in \mathcal{G}_2 such that two vertices in L are adjacent if the corresponding hyperedges in \mathcal{G}_2 share a pair. Then L has maximum degree at most 2 and thus has an independent set of size at least $n(L)/3$. Therefore, there is a linear subhypergraph \mathcal{G}_2^* of \mathcal{G}_2 with $|\mathcal{G}_2^*| \geq \frac{1}{3}|\mathcal{G}_2|$. Hence $d(\mathcal{G}_2^*) \geq 21(k + 1)$. By Theorem 3.4, \mathcal{G}_2^* contains Berge cycles of k consecutive lengths. \square

We now prove Theorem 1.3.

Proof of Theorem 1.3. We prove by induction on $r \geq 3$ that every r -graph \mathcal{H} with $d(\mathcal{H}) \geq r \cdot (35k^{r-1} + 21k^{r-2})$ contains Berge cycles of k consecutive lengths. Theorem 4.4 forms the basis step. For the induction step, let $r \geq 4$.

Assume the claim holds for $(r-1)$ -graphs. Let \mathcal{H} be an r -graph with $d(\mathcal{H}) \geq r \cdot (35k^{r-1} + 21k^{r-2})$. By Lemma 4.1, either there exists a subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with $\delta_{r-1}(\mathcal{H}') \geq k+1$, or there exists an extendable $(r-1)$ -graph $\mathcal{G} \subseteq \partial_{r-1}(\mathcal{H})$ such that $|\mathcal{G}| \geq |\mathcal{H}|/k$. In the former case, by Lemma 4.3, we can find Berge cycles of lengths in $\{3, 4, \dots, k+2\}$ in \mathcal{H} and we are done. So assume the latter case. Then $d(\mathcal{G}) \geq \frac{r-1}{r} \cdot \frac{d(\mathcal{H})}{k} \geq (r-1) \cdot (35k^{r-2} + 21k^{r-3})$. By induction, \mathcal{G} contains Berge cycles of k consecutive lengths. Because \mathcal{G} is extendable, \mathcal{H} also contains Berge cycles of the same k consecutive lengths. \square

We then have the following corollary.

Corollary 4.5. *For all $r \geq 3$, any n -vertex r -graph \mathcal{H} with at least $56k^{r-1}n$ edges contains Berge cycles of k consecutive lengths.*

5. Related Turán type results

5.1. Cycles of consecutive even lengths in graphs

Following arguments along the lines of Lemma 3.2 (i.e., to define *monochromatic* and *non-monochromatic* edges and then apply Lemma 2.6), we can readily prove the following slightly weaker version of Verstraëte's theorem.

Proposition 5.1. *Let G be a bipartite graph with average degree at least $6k$ and radius h . Then G contains cycles of k consecutive even lengths. Further, the shortest of these cycles has length at most $2h$.*

This provides a first proof which does not use the (A, B) -path lemma (Lemma 1.5). This also gives yet another proof of the theorem of Bondy and Simonovits on $ex(n, C_{2k})$ without using either the (A, B) -path lemma or the Faudree-Simonovits blowup method. As Lemma 2.6 (and its hypergraph extension) can be easily adapted, we anticipate this new method will find further applications in Turán type extremal problems on cycles in graphs or hypergraphs.

5.2. Berge cycles of prescribed consecutive lengths

Let BC_ℓ denote the family of r -graphs consisting of all Berge cycles of length ℓ . Let $r \geq 3$ and \mathcal{H} be an r -graph with n vertices. Corollary 4.5 shows that if $|\mathcal{H}| \geq 56k^{r-1}n$,

then \mathcal{H} contains Berge cycles of k consecutive lengths. If in addition to find Berge cycles of k consecutive lengths one also wants to control the lengths to not be large, i.e., the maximum length is no more than $k + p$, then how many edges in an r -graph will suffice? In this subsection we provide an answer to this question (see Theorem 5.3).

Using Lemmas 3.1 and 3.2, we can prove the following theorem for linear 3-graphs. Its proof uses similar arguments as the ones in [18] by Füredi and Özkahya, who proved that any n -vertex \mathcal{BC}_{2k+1} -free linear 3-graph has at most $2k \cdot n^{1+1/k} + 9k \cdot n$ edges. However their proof method was not designed for finding Berge cycles of consecutive lengths.

Theorem 5.2. *Let $h, k \geq 2$. Every n -vertex linear 3-graph \mathcal{H} with $|\mathcal{H}| \geq 18kn^{1+1/h} + 42kn$ contains Berge cycles of k consecutive lengths, the shortest of which has length at most $2h$.*

Proof. Suppose for a contradiction that \mathcal{H} does not contain Berge cycles of k consecutive lengths, the shortest being at most $2h$. By Lemma 2.2, there exists a subhypergraph \mathcal{H}' of \mathcal{H} with $\delta(\mathcal{H}') \geq 18kn^{1/h} + 42k$. Let T be a maximal extendable skeleton in \mathcal{H}' rooted at some vertex r . For each $i \geq 0$, let $L_i = \{v \in V(T) : d_T(r, v) = i\}$, $\mathcal{A}_i = \{e \in \mathcal{H}' : |e \cap L_i| = 2, |L_{i-1}| = 1\}$, $\mathcal{B}_i = \{e \in \mathcal{H}' : |e \cap L_i| = 2, |e \cap L_{i+1}| = 1\}$, and $\mathcal{C}_i = \{e \in \mathcal{H}' : e \subseteq L_i\}$. By Lemmas 3.1 and 3.2, for any $i \leq h - 1$ we may assume (being quite generous for the sake of simplicity) that

$$|\mathcal{A}_i| \leq 2k|L_i| \text{ and } |\mathcal{B}_i \cup \mathcal{C}_i| \leq 4k|L_i| + 4k|L_{i+1}|. \tag{6}$$

We prove by induction on $1 \leq i \leq h - 1$ that $|L_i| \geq |L_{i-1}| \cdot n^{1/h}$. The base case $i = 1$ follows by the facts that \mathcal{H}' is linear and $\delta(\mathcal{H}') \geq 18kn^{1/h}$. Now suppose that it holds for $i \leq h - 2$. We consider the number m of edges intersecting L_i . We have $m = |\mathcal{A}_i| + |\mathcal{B}_i| + |\mathcal{C}_i| + |\mathcal{C}_{i-1}| + |\mathcal{A}_{i+1}|$. By (6),

$$\begin{aligned} m &\leq 2k|L_i| + 4k|L_i| + 4k|L_{i+1}| + 4k|L_{i-1}| + 4k|L_i| + 2k|L_{i+1}| \\ &= 4k|L_{i-1}| + 10k|L_i| + 6k|L_{i+1}| \leq 14k|L_i| + 6k|L_{i+1}|. \end{aligned}$$

On the other hand, $m \geq \frac{1}{3} \cdot \sum_{v \in L_i} d_{\mathcal{H}'}(v) \geq |L_i| \cdot (6kn^{1/h} + 14k)$. Combining the above inequalities, it follows that $|L_{i+1}| \geq |L_i| \cdot n^{1/h}$. Therefore, $|L_h| \geq n$. This contradiction completes the proof. \square

We point out that just like in the proof of Theorem 1.4, one can establish a similar statement for linear r -graphs for all $r \geq 3$ (by reducing them to linear 3-graphs).

Remark. Observe that the proof of Theorem 5.2 in fact yields the following more general statement: If \mathcal{H} is an n -vertex linear 3-graph with average degree $d \geq 45k$, then \mathcal{H} contains Berge cycles of k consecutive lengths, the shortest of which has length at most $O(\log_{d/k} n)$.

Using the reduction lemmas in Section 4, along the same lines as in the proof of Theorem 1.3 (and Theorem 4.4), we also can obtain the following result from Theorem 5.2. We omit the details.

Theorem 5.3. *There exists an absolute constant $c > 0$ such that the following holds for all $h, k \geq 2$ and $r \geq 3$. Every n -vertex r -graph \mathcal{H} with at least $ck^{r-1}n^{1+1/h}$ edges contains Berge cycles of k consecutive lengths, the shortest of which has length at most $2h$.*

When choosing $k = 2h$, this may be viewed as an unification for the results on $ex_r(n, \mathcal{BC}_{2h})$ and $ex_r(n, \mathcal{BC}_{2h+1})$.

5.3. Turán numbers of Berge cycles in r -graphs

In this subsection we investigate the upper bounds of Turán numbers $ex_r(n, \mathcal{BC}_\ell)$ for $r \geq 3$. We start by mentioning the Turán numbers of even cycles in the graph case. A classic theorem of Bondy and Simonovits [4] shows that $ex(n, C_{2k}) \leq 100k \cdot n^{1+1/k}$, and this bound was improved by several authors in [37,31,6]. The current best known upper bound is the following one obtained by Bukh and Jiang [6]

$$ex(n, C_{2k}) \leq 80\sqrt{k \log k} \cdot n^{1+1/k} + O(n). \tag{7}$$

For 3-graphs, Győri and Lemons proved that $ex_3(n, \mathcal{BC}_{2k+1}) \leq O(k^4) \cdot n^{1+1/k}$ in [21] and that $ex_3(n, \mathcal{BC}_{2k}) \leq O(k^2) \cdot ex(n, C_{2k})$ in [22]. Füredi and Özkahya [18] improved this by showing that

$$ex_3(n, \mathcal{BC}_{2k+p}) \leq O(k) \cdot ex(n, C_{2k}) + 12p \cdot ex_3^{lin}(n, \mathcal{BC}_{2k+1}) \quad \text{for every } p \in \{0, 1\},$$

where $ex_r^{lin}(n, \mathcal{F})$ denotes the maximum number of hyperedges in an n -vertex \mathcal{F} -free linear r -graph and it is also proved in [18] that $ex_3^{lin}(n, \mathcal{BC}_{2k+1}) \leq 2k \cdot n^{1+1/k} + 9k \cdot n$. (See [1] for related problems.) In view of (7), one can obtain

$$ex_3(n, \mathcal{BC}_{2k+p}) \leq O(k\sqrt{k \log k}) \cdot n^{1+1/k} \quad \text{for every } p \in \{0, 1\}. \tag{8}$$

In [22], Győri and Lemons also showed for general r -graphs, where $r \geq 4$, that

$$ex_r(n, \mathcal{BC}_{2k+1}) \leq O_r(k^{r-2}) \cdot ex_3(n, \mathcal{BC}_{2k+1}), \tag{9}$$

$$ex_r(n, \mathcal{BC}_{2k}) \leq O_r(k^{r-1}) \cdot ex(n, C_{2k}). \tag{10}$$

Using the lemmas in Section 4, one can also derive some Turán type results on Berge cycles, which improve the above inequalities (9) and (10) by an $\Omega(k)$ factor.

Proposition 5.4. *For all $r \geq 4$, it holds that*

$$ex_r(n, \mathcal{BC}_{2k+1}) \leq (2k)^{r-3} \cdot ex_3(n, \mathcal{BC}_{2k+1})$$

$$ex_r(n, \mathcal{BC}_{2k}) \leq (2k)^{r-2} \cdot ex(n, C_{2k}).$$

Therefore for any $p \in \{0, 1\}$,

$$ex_r(n, \mathcal{BC}_{2k+p}) \leq O_r(k^{r-2} \sqrt{k \log k}) \cdot n^{1+1/k}.$$

The proof of Proposition 5.4 follows easily from Lemmas 4.1 and 4.3. We omit the details. We mention another related result. In [22], the following result was also proved for non-uniform hypergraphs: for any $p \in \{0, 1\}$, if \mathcal{H} is a multi-hypergraph on n vertices with all of its hyperedges of size at least $4k^2$ and containing no Berge cycle of length $2k + p$, then

$$\sum_{e \in \mathcal{E}(\mathcal{H})} |e| \leq (16k^6 + 8k^2) \cdot n^{1+1/k} + (16k^7 + 32k^6 + 16k^5) \cdot n. \tag{11}$$

5.4. Asymmetric Turán numbers of even cycles in graphs

Let the Zarankiewicz number $z(m, n, C_{2k})$ of the even cycle C_{2k} to be the maximum number of edges in a C_{2k} -free bipartite graph with two parts of sizes m and n . An upper bound was proved by Naor and Verstraëte [30] that for $m \leq n$ and $k \geq 2$,

$$z(m, n, C_{2k}) \leq \begin{cases} (2k - 3) \cdot [(mn)^{\frac{k+1}{2k}} + m + n] & \text{if } k \text{ is odd,} \\ (2k - 3) \cdot [m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n] & \text{if } k \text{ is even.} \end{cases} \tag{12}$$

In this subsection, we consider a different form of upper bounds on $z(m, n, C_{2k})$.

Erdős, Sárközy and Sós [12] conjectured that $z(m, n, C_6) < 2n + c(nm)^{2/3}$ for some constant $c > 0$. A weaker version of this conjecture was obtained by Sárközy in [32]. Györi [19] proved a general result: there exists some $c_k > 0$ such that for $n \geq m^2$,

$$z(m, n, C_{2k}) \leq (k - 1)n + c_k \cdot m^2. \tag{13}$$

The first term $(k - 1)n$ is sharp (at least in a sense) by considering the complete bipartite graph $K_{k-1, n}$; and when $n = \Omega(m^2)$ this function becomes linear in n . Some related results also can be found in [2, 20].

The following upper bound of $z(m, n, C_{2k})$ can be derived from the Turán numbers of Berge cycles in hypergraphs, which is stronger than (13).

Proposition 5.5. *There exists a constant $d_k > 0$ such that for any positive integers n, m ,*

$$z(m, n, C_{2k}) \leq (k - 1)n + d_k \cdot m^{1+1/\lfloor k/2 \rfloor}.$$

Proof. Let G be any bipartite C_{2k} -free graph with two parts A and B , where $|A| = m$ and $|B| = n$. Define \mathcal{H}_0 and \mathcal{H}_i for every $k \leq i < k^2$ to be multi-hypergraphs with the

vertex-set A such that $\mathcal{E}(\mathcal{H}_0) = \{N_G(u) : d_G(u) \geq k^2 \text{ for } u \in B\}$ and $\mathcal{E}(\mathcal{H}_i) = \{N_G(u) : d_G(u) = i \text{ for } u \in B\}$. Then

$$e(G) = \sum_{u \in B} d_G(u) \leq (k - 1)n + \sum_{e \in \mathcal{E}(\mathcal{H}_0)} |e| + \sum_{k \leq i < k^2} \sum_{e \in \mathcal{E}(\mathcal{H}_i)} |e|.$$

Observe that \mathcal{H}_0 and all \mathcal{H}_i do not contain Berge cycle of length k , as otherwise it will give a C_{2k} in G . Since \mathcal{H}_0 has no Berge cycle of length k and each hypergraph of \mathcal{H}_0 is of size at least $k^2 \geq 4(\lfloor k/2 \rfloor)^2$, by the equation (11), one can get that

$$\sum_{e \in \mathcal{E}(\mathcal{H}_0)} |e| \leq O(k^6) \cdot m^{1+1/\lfloor k/2 \rfloor}.$$

Consider the multi-hypergraph \mathcal{H}_i , where $k \leq i < k^2$. It is easy to see that there are at most $k - 1$ hyperedges which are identical (otherwise we have $C_{2k} \subseteq K_{k,k} \subseteq G$). Thus, there exists a simple \mathcal{BC}_k -free i -graph $\mathcal{H}'_i \subseteq \mathcal{H}_i$ such that $|\mathcal{H}'_i| \geq |\mathcal{H}_i|/k$. By Proposition 5.4 (or Theorem 5.3), $|\mathcal{H}_i| \leq k \cdot |\mathcal{H}'_i| \leq O(k^i) \cdot m^{1+1/\lfloor k/2 \rfloor}$. Combining the above inequalities, one can obtain that

$$e(G) \leq (k - 1)n + O(k^6) \cdot m^{1+1/\lfloor k/2 \rfloor} + \sum_{k \leq i < k^2} i \cdot |\mathcal{H}_i| \leq (k - 1)n + d_k \cdot m^{1+1/\lfloor k/2 \rfloor},$$

where $d_k = O(k^{k^2+1})$. This finishes the proof. \square

One may compare this proposition with (12). In the range $m^{1+1/\lfloor k/2 \rfloor}/n \rightarrow 0$, interestingly both upper bounds become linear in n . To be precise, (12) gives that $z(m, n, C_{2k}) \leq (2k - 3 + o(1)) \cdot n$, and this proposition yields $z(m, n, C_{2k}) \leq (k - 1 + o(1)) \cdot n$, which is nearly tight.

6. Concluding remarks

6.1. Finding tight conditions for Berge cycles of consecutive lengths

It will be interesting to completely solve the problem on Berge cycles of consecutive lengths.

Problem 6.1. For $r \geq 3$, find the minimum $f_r(k)$ such that any r -graph with average-degree $f_r(k)$ (or minimum-degree) contains Berge cycles of k consecutive lengths.

Note that Theorem 1.3 implies that $f_r(k)$ exists as a finite number. The complete r -graph on $k + 1$ vertices shows that $f_r(k) > \binom{k}{r-1}$. One also can ask for the tight degree condition for the existence of Berge cycles of k consecutive lengths in linear r -graphs.

6.2. Linear cycles of consecutive lengths and the linear Turán problem

It is natural to consider the analogous problem for forcing linear cycles (rather than Berge cycles) of consecutive lengths in an r -graph. When the host graph G is a general r -graph, we need $\Omega(n^{r-1})$ edges to force any linear cycle at all. Indeed, the full star on $[n]$ contains $\binom{n-1}{r-1}$ hyperedges and no linear cycle. So any average degree condition on G to ensure the existence of linear cycles of k consecutive lengths has to involve n , the number of vertices. It may therefore be more natural to cast this as a Turán problem. Note that the Turán problem for a single linear cycle of a fixed length has been precisely solved for large n by Füredi and Jiang [17] (for $r \geq 5$) and by Kostochka, Mubayi and Verstraëte [26] (for $r \geq 3$). Asymptotically, $ex_r(n, C_{2k+p}) \sim (k + p - 1) \cdot \binom{n}{r-1}$, where $p \in \{0, 1\}$.

One may also consider both the Turán problems of a single linear cycle of a fixed length and the linear cycles of k consecutive lengths in linear r -graphs. The former problem was studied by Collier-Cartaino, Graber and Jiang [7]. Let $ex_r^{lin}(n, C_\ell)$ denote the maximum number of hyperedges in an n -vertex linear r -graph that does not contain a linear cycle of length ℓ . Extending Bondy–Simonovits [4], the authors of [7] showed that $ex_r^{lin}(n, C_{2k+p}) \leq c_{r,k} n^{1+1/k}$ for some constant $c_{r,k} > 0$ depending on r and k , where $p \in \{0, 1\}$. Concerning the problem of forcing linear cycles of some k consecutive lengths in a linear r -graph, it is natural to ask if there exists a constant $\alpha_{r,k}$ depending only on r and k such that every linear r -graph with average degree at least $\alpha_{r,k}$ contains linear cycles of k consecutive lengths. Very recently, building on the method developed in this paper, some ideas in [7] and some new ideas, the authors of [24] showed that this is indeed the case. Specifically, they proved that every linear r -graph with average degree $\Omega_r(k)$ contains linear cycles of k consecutive lengths. This strengthens Theorem 1.4 (except for the constant). The authors of [24] were also able to improve the constant $c_{r,k}$ in the upper bound of $ex_r^{lin}(n, C_{2k})$ to $O_r(k)$.

6.3. Rainbow Turán problem for even cycles

A problem that is closely related to the linear Turán problem of linear cycles is rainbow Turán problem for even cycles. For a fixed graph H , define the *rainbow Turán number* $ex^*(n, H)$ to be the maximum number of edges in an n -vertex graph that has a proper edge-coloring with no rainbow H . Keevash, Mubayi, Sudakov and Verstraëte [25] made the following conjecture

Conjecture 6.2. ([25]) *For all $k \geq 2$, $ex^*(n, C_{2k}) = O(n^{1+1/k})$.*

The conjecture was verified for $k = 2, 3$ in [25], but is otherwise still open. It will be interesting to see if the method developed here can be used to make some progress on the problem.

We direct readers to the recent survey [38] by Verstraëte for various other extremal problems on cycles.

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